EQUITY AND CONTINUITY WITH A CONTINUUM OF GENERATIONS

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Abstract

In this paper we consider social preferences over intergenerational consumption paths for a continuum of generations. We prove that equal treatment of all generations is incompatible with Mackey continuity of social preferences under mild axioms such as nontriviality and asymmetry. We also show that this impossibility result is tight in the sense that for each of the four axioms there exists a social preference relation violating it but satisfying all the other axioms.

1. Introduction

In evaluating each resource allocation over time, a central planner needs to employ a social welfare function or, more generally, a social preference relation defined on the set of possible resource allocations. In his seminal paper on optimal saving in a continuous time model, Ramsey (1928) suggested to construct a particular social welfare function by integrating undiscounted utility streams over time. Ramsey was against discounting future utilities since it fails to treat all generations equally. While intergenerational equity may be quite appealing from an ethical point of view, the resulting social preference relation does not satisfy continuity in any sensible topologies. This could cause a serious technical problem for the existence of optimal capital accumulation. (See, for example, Romer (1986).) Thus we face a trade-off between intergenerational equity and continuity.

This trade-off attracted the attention of researchers. Diamond (1965) showed in a discrete time setting that continuity in the product topology imposes a certain form of impatience on social preference orderings, which violates intergenerational equity. This impossibility result of Diamond's leads Svensson (1980) to find a certain larger topology compatible with equity. Campbell (1985) critically re-examined the Svensson topology to propose another topology for generalizing Diamond's impossibility theorem. Lauwers (1997) discusses relative merits of prominent linear topologies. Shinotsuka (1998) focuses on the Mackey topology to obtain a yet another generalization of Diamond's impossibility theorem.

So far, contributions to this equity-continuity trade-off have been confined to discrete time models. And yet, we believe the continuous setting merits study. For one thing, it offers a direct account for the equity-continity trade-off observed in the continuous time growth models. Discrete time models exhibit special features. First, in l_∞ space, the strict topology and the Mackey topology $\tau(l_\infty, l_1)$ coincides (see Conway (1966).). Shinotsuka (1998) used this fact to show that a Mackey continuous preference relation "in general" violates equity. Second, in $\mathbb{R}^\infty$ space, product convergence is equivalent to order convergence (see Becker and Boyd (1997, Chapter 2, Theorem 9), for example). This fact allows us to develop an alternative approach to a generalization of Diamond's impossibility theorem (Shinotsuka (1998)). Thus, it is only natural to investigate to what extent one could generalize arguments developed in a discrete time setting to the continuous case. The purpose of this paper is to formulate this trade-off from an axiomatic perspective in the presence of a continuum of generations. We formulate equity in pretty much the same way Kaneko (1981, 1984) did for the axiomatic theory of the Nash social welfare function for a continuum of individuals. The intuitive content of equity is simply this: social preferences should be independent of the names of generations.[3)]

We also discuss the crucial role of asymmetry on the part of social preferences. If we do not insist on asymmetry, then we can construct a social preference relation satisfying non-triviality, equity, and Mackey continuity. Together with the foregoing impossibility theorem, this tells the following: Equity and Mackey continuity are compatible but only in the case of unreasonable preferences admitting "two-term cycles". A corollary: There is no non-constant Bergson-Samuelson social welfare function satisfying equity and Mackey continuity.

We now briefly review the use of Mackey topology in economic theory. Bewley (1972) introduced the Mackey topology into general equilibrium theory and argued that Mackey topology is appropriate for dated or state-contingent commodities. Brown-Lewis (1981) formalized Bewley's intuition by characterizing Mackey topology on l_∞ by means of myopic topologies. Araujo (1985) showed that Mackey continuity of preferences are necessary for the existence of competitive equilibrium in l_∞. Raut (1986) extended the results of Brown-Lewis and Araujo from l_∞ to $\mathscr{L}_\infty$. In the context of efficiency prices, Majumdar (1972) used Mackey topology to obtain a price vector in $\mathscr{L}_1$ which supports a plan in $\mathscr{L}_\infty$ maximizing quasi-concave, increasing, and Mackey continuous social welfare function. Kobayashi (1980) made an extensive use of Mackey topology to prove the existence of a constrained equilibrium contract for syndicates with differential information. Takekuma (1980) used Mackey topology to conduct a sensitivity analysis on optimal economic growth in a continuous time model of capital accumulation.

The next section provides preliminaries and the axioms. Section three gives statements of a lemma and two theorems. Section four collects the proofs of them. The last section includes an open problem.

2. Concepts and axioms

We consider a measure space of generations (T, A, μ), where T is the set of all nonnegative real numbers, A is the σ–algebra of Lebesgue measurable subsets of T, and μ is the Lebesgue measure on T. Each member of T signifies a particular generation. L_∞ is the set of all essentially bounded, real-valued, Lebesgue measurable functions on (T, A, μ) and L_1 is the set of all integrable real-valued functions on (T, A, μ). We call each member of L_∞ a consumption path.[4]

A social preference relation $\succ$ is a binary relation on L_∞. The intended interpretation is that a consumption path x is socially preferred to another y if and only if $x \succ y$ holds. With the measure-theoretic structure placed on the set T of generations, it seems natural to require a social preference relation be invariant with respect to any change in consumption path on any subset of μ-measure zero. We claim that this is not only a matter of mathematical convenience but also a sort of consistency requirement: It would be puzzling if the evaluation of resource allocations by the central planner depended on how to assign consumption to generations which, as a group, have no influence whatsoever to the whole economy.
We regard two consumption paths x and x' identical if $\mu(\{t \in T \mid x(t) \neq x'(t)\}=0$. We write $x \cong x'$ if $\mu(\{t \in T \mid x(t) \neq x'(t)\})=0$. $\cong$ is an equivalence relation. Let $\mathscr{L}_\infty$ (resp. $\mathscr{L}_1$) be the set of all equivalence classes of L_∞ (resp. L_1) under the relation $\cong$.

From equivalence, we may regard $\mathscr{L}_\infty$ as the set of consumption paths and a binary relation $\succ$ on $\mathscr{L}_\infty$ as a social preference relation. Generic elements in $\mathscr{L}_\infty$ are denoted by bold face letters $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, e. t. c. A representative element of $\mathbf{x}$ is denoted by x, and so on.

When we refer to a measure-preserving automorphism on T, it always mean a bijective measurable function π from (T, A, μ) onto itself such that the inverse function of it is also measurable and $\mu(S)=\mu(\pi(S))$ for all $S \in \mathsf{A}$. A measure-preserving automorphism on T represents a rule of how to rearrange hypothetically the chronological order of generations. A measure-preserving automorphism π is finite if there exists t_0 such that $\pi(t)=t$ for all $t \geq t_0$. That is, in a finite measure-preserving automorphism, the rearrangement of generations occurs only in an "initial segment" of generation interval T. A social preference $\succ$ is equitable if for any consumption paths $\mathbf{x}$ and $\mathbf{y}$, and for any finite, measure-preserving automorphism π on T, $\mathbf{x} \succ \mathbf{y}$ implies $\mathbf{x} \circ \pi \succ \mathbf{y} \circ \pi$.[5] When the last implication holds for an arbitrary measure-reserving automorphism, we say that the social preference relation is strongly equitable. Strong equity requires the social preference relation treats all generations equally. Although strong equity appears more appealing as an axiom, we focus on equity: Since our main goal is to establish an impossibility theorem, a weaker axiom should suffice.[6]

We use the Mackey topology $\tau(\mathscr{L}_\infty, \mathscr{L}_1)$ to describe continuity of social preference relations. Recall that the Mackey topology $\tau(\mathscr{L}_\infty, \mathscr{L}_1)$ is generated by a family of semi-norms of the following form: $p_C(\mathbf{x}) = \sup\{|\int \mathbf{x} \cdot \mathbf{y} d\mu| : \mathbf{y} \in C\}$. for $\mathbf{x} \in \mathscr{L}_\infty$, where $\mathbf{x} \cdot \mathbf{y}$ denotes the product of $\mathbf{x}$ and $\mathbf{y}$ defined by $(\mathbf{x} \cdot \boldsymbol{y})(t) = x(t) \cdot y(t)$ for $t \in T$, and C is an arbitrary $\sigma(\mathscr{L}_1, \mathscr{L}_\infty)$-compact, convex, circled subset of $\mathscr{L}_1$. When we say the Mackey topology in the subsequent discussions, it always means $\tau(\mathscr{L}_\infty, \mathscr{L}_1)$.

We list our axioms.

Nontriviality: There exist consumption paths $\mathbf{x}$ and $\mathbf{y}$ with $\mathbf{x} \succ \mathbf{y}$.

Nontriviality is an extremely weak requirement in that the only social preference relation violating this axiom is the one that regards all pairs of consumption paths non-comparable.

The following axiom is standard in social choice theory.

Asymmetry: For all $\mathbf{x}$ and $\mathbf{y}$ in $L_\infty^+ L$, $\mathbf{x} \succ \mathbf{y}$ implies not $\mathbf{y} \succ \mathbf{x}$.

The following axiom requires that a social preference relation treats all generations equally.

Equity: $\succ$ is equitable.

The following axiom requires that a social preference relation is continuous with respect to the Mackey topology.

Mackey Continuity: $\succ$ is $\tau(\mathscr{L}_\infty, \mathscr{L}_1) \times \tau(\mathscr{L}_\infty, \mathscr{L}_1)$-open.

3. Statement of Theorems

The following impossibility theorem is the main result of this paper.

Theorem 1: There is no social preference relation on $\mathscr{L}_\infty$ satisfying non-triviality, asymmetry, equity, and Mackey continuity.

The next result says that if we drop any one of the axioms in Theorem 1, we have a possibility result. Thus the impossibility result in Theorem 1 is tight.

Theorem 2: For each axiom in Theorem 1, there exists a social preference relation on $\mathscr{L}_\infty$ violating it but satisfying all the other axioms.

4. Proofs

Lemma 1: Let $\{\mathbf{x}^n\}$ be a sequence in $\mathscr{L}_\infty$ converging to $\mathbf{x}$ in the Mackey topology, and let $\{\mathbf{y}^n\}$ be a norm-bounded sequence in $\mathscr{L}_\infty$. Let x^n, y^n be representative elements of $\mathbf{x}^n$ and $\mathbf{y}^n$, respectively. Let $z^n(t)=x^n(t)$ if $t\in[0, n]$ and $z^n(t)=y^n(t-n)$, otherwise. Let z^n be an equivalence class with a representative element z^n. Then $\{z^n\}$ converges to x in the Mackey topology.

Proof of Lemma 1: Let C be a $\sigma(\mathscr{L}_1, \mathscr{L}_\infty)$-compact, convex, circled subset of $\mathscr{L}_1$. We have to show sup $\{|\int(z^n-\mathbf{x})\cdot vd\mu|:v\in C\}$ converges to zero as n goes to $+\infty$. By the Eberlein-Šmulian Theorem（Dunford-Schwartz（1958, p.430 Theorem 1.)), C is sequentially $\sigma(\mathscr{L}_1, \mathscr{L}_\infty)$-compact. Clearly, we have

$$|\int(z^n-\mathbf{x})\cdot vd\mu|\leq|\int(x^n(t)-x(t))\cdot vd\mu|+|\int_{[n,\infty)}(y^n(t-n)-x^n(t))\cdot v(t)d\mu(t)|$$

By assumption, the first term in the right hand side converges to zero uniformly in $v\in C$. Therefore, it is sufficient to show that the second term converges to zero uniformly in $v\in C$.

The sequence $\{y^n\}$ is norm-bounded by assumption, and the sequence $\{x^n\}$ is also norm-bounded since it is Mackey convergent. Hence for each $v\in L_1$, $\{\int x^n(t)\cdot v(t)d\mu(t)\}$ converges to $\int x(t)\cdot v(t)d\mu(t)$. Hence for each $v\in L_1$, $\{\int x^n(t)\cdot v(t)d\mu(t)\}$ is bounded. By the Uniform Boundedness Principle（Dunford-Schwartz（1958, Ⅱ.3.27, p.68), $\{x^n\}$ is norm-bounded. With these norm- bounded sequences in hand, let us evaluate the term:

$$|\int_{[n,\infty)}(y^n(t-n)-x^n(t))\cdot v(t)d\mu(t)|\leq\int_{[n,\infty)}|v(t)|d\mu(t)\cdot\|(y^n-x^n)\|\leq\int_{[n,\infty)}|v(t)| d\mu(t)\cdot(\sup_n\|y^n\|_\infty-\sup_n\|x^n\|_\infty)$$

By Dunford-Schwartz［1958, p. 293, Corollary 10, and p. 292, Theorem 9］, the integral $\int_{[n,\infty]}|v(t)|d\mu(t)\}$ converges to zero as $n\to+\infty$, uniformly in $v\in C$. This completes the proof.

■

Proof of Theorem 1: Suppose there exists a social preference relation $\succ$ on $\mathscr{L}_\infty$ satisfying the four axioms. By non-triviality, there exist consumption paths $\mathbf{x}$ and $\mathbf{y}$ such that $\mathbf{x} \succ \mathbf{y}$. Let $n \in \mathbb{N}$. Define two sequences $\{\mathbf{x}^n\}$, $\{\mathbf{y}^n\}$ of consumption paths as follows. Let $\mathbf{x}$ and $\mathbf{y}$ be representative elements of $\mathbf{x}$ and $\mathbf{y}$, respectively. Let $x^n(t)=x(t)$ if $t \in [0, n]$ and let $x^n(t)=y(t-n)$ otherwise. Let $y^n(t)=y(t)$ if $t \in [0, n]$ and let $y^n(t)=x(t-n)$ otherwise. Let $\mathbf{x}^n$ and $\mathbf{y}^n$ be equivalence classes with representative elements x^n and y^n, respectively. By Lemma 1, $\{\mathbf{x}^n\}$ and $\{\mathbf{y}^n\}$ converge to $\mathbf{x}$ and $\mathbf{y}$, respectively, in the Mackey topology. By Mackey continuity, there exists a positive real number N such that $\mathbf{x}^n \succ \mathbf{y}^n$ for all $n \geq N$. Let fix such an N.

For each $m \in \mathbb{N}$, define a finite measure-preserving isomorphism π^m on T by the following. Let $\pi^m(t)=t+N$ if $t \in [(m-1)N, mN]$, let $\pi^1(t)=t-N$ if $t \in [mN, (m+1)N)$, and $\pi^1(t)=t$, otherwise. Consider two sequences $\{v^m\}$ and $\{w^m\}$ defined by

$$v^m = (\underbrace{\cdots(x^N \circ \pi^1)\cdots}_{m \text{ times}}) \circ \pi^m, v^m = (\underbrace{\cdots(y^N \circ \pi^1)\cdots}_{m \text{ times}}) \circ \pi^m$$

Let $\mathbf{v}^m$ and $\mathbf{w}^m$ be equivalence classes with representative elements v^m and w^m, respectively. By equity, we have $\mathbf{v}^m \succ \mathbf{w}^m$ for all m.

It is easy to see $v^m(t)=y(t)$ for all $t \in [0, mN)$ and $w^m(t)=x(t)$ for all $t \in [0, mN)$. We also have

$$||\mathrm{v}^m||_\infty \leq \max\{||\mathrm{x}^m||_\infty, ||\mathrm{y}^m||_\infty\} \text{ and } ||\mathrm{w}^m||_\infty \leq \max\{||\mathrm{x}^m||_\infty, ||\mathrm{y}^m||_\infty\} \text{ for all } m.$$

Hence, by Lemma 1, we can conclude that $\{v^m\}$ and $\{w^m\}$ converge to $\mathbf{y}$ and $\mathbf{x}$, respectively, in the Mackey topology. By Mackey continuity, there exists some positive real number M such that $\mathbf{w}^m \succ \mathbf{v}^m$ for all $m \geq M$. Hence, for $m \geq M$ we have both $\mathbf{v}^m \succ \mathbf{w}^m$ and $\mathbf{w}^m \succ \mathbf{v}^m$. A contradiction to asymmetry of $\succ$. This completes the proof. ■

Proof of Theorem 2: We exhibit four examples some of which are well-known. Example 2 is new.

Example 1. Let $\succ = \phi$. Then $\succ$ violates nontriviality but satisfies the other axioms.

Example 2. Define $\succ$ by $\mathbf{x} \succ \mathbf{y}$ if and only if there exists $A \in \beta(T)$ such that $0 < \mu(A) < \infty$ and $\int_A \mathbf{x} d\mu > \int_A \mathbf{y} d\mu$. Clearly, $\succ$ violates asymmetry. In order to show Mackey continuity, suppose $0 < \mu(A) < \infty$ and $\int_A \mathbf{x} d\mu > \int_A \mathbf{y} d\mu$. Let $\boldsymbol{\chi}_A$ be the indicator function of the set A and let

$C=\{\alpha\chi_A \mid \alpha\in\mathcal{R}, |a|\leq 1\}$. C is trivially non-empty, $\sigma(\mathcal{L}_1, \mathcal{L}_\infty)$-compact, convex, and circled subset of L_1. Let $q_C(\mathbf{x})=\sup\{|\int p\cdot\mathbf{x}d\mu| : p\in C\}=|\int_A \mathbf{x}d\mu|$.

By definition of the Mackey topology $\tau(\mathcal{L}_\infty, \mathcal{L}_1)$. $qc(\cdot)$ is one of the semi-norms generating it. Let ε be a positive real number such that $\int_A \mathbf{x}d\mu > \int_A \mathbf{y}d\mu+\varepsilon$. Let $V(x)$ = $\{\mathbf{z}\in\mathcal{L}_\infty \mid q_k(\mathbf{z}-\mathbf{x})<\varepsilon/2\}$ $=\{\mathbf{z}\in\mathcal{L}_\infty \mid |\int_A(\mathbf{z}-\mathbf{x})d\mu|<\varepsilon/2\}$. Define $V(\mathbf{y})$ in the same way. It follows from the foregoing discussion that $V(\mathbf{x})$ and $V(\mathbf{y})$ are Mackey neighborhoods of $\mathbf{x}$ and $\mathbf{y}$, respectively. We show that $\mathbf{x}'\in V(\mathbf{x})$ and $\mathbf{y}'\in V(\mathbf{y})$ imply $\mathbf{x}'\succ\mathbf{y}'$. By a simple computation, we obtain

$$\int_A \mathrm{x}'\, d\mu-\int_A \mathrm{y}'\, d\mu>\left(\int_A \mathrm{x}d\mu-\frac{\varepsilon}{2}\right)-\frac{\varepsilon}{2}-\int_A \mathrm{y}d\mu>0.$$

Hence $\mathbf{x}'\succ\mathbf{y}'$. Therefore, $\succ$ is Mackey continuous. It is straightforward to check that $\succ$ satisfy equity.

Example 3. Let u: $\mathbb{R}\to\mathbb{R}$ be a continuous, strictly increasing function with $u(0)=0$. Let $U{:}L_\infty^+\to\mathbb{R}$ be a utility function defined by $U(\mathbf{x})=\int\mathbf{x}(t)e^{-t}d\mu(t)$. By Bewley [1972, Appendix II], U is Mackey continuous. Define $\succ$ by $\mathbf{x}\succ\mathbf{y}$ if and only if $U(\mathbf{x})>U(\mathbf{y})$. Clearly, $\succ$ violates equity but satisfies the other axioms.

Example 4. Define $\succ$ by the following. $\mathbf{x}\succ\mathbf{y}$ if and only if there exists a positive real number t^0 such that

$$\int_{[0,t']}\mathrm{x}(t)d\mu(t)>\int_{[0,t']}\mathrm{y}(t)d\mu(t) \text{ for all } t'\geq t^0.$$

Then, $\succ$ violates Mackey continuity. To see this, let $\mathbf{x}(t)=2e^{-t}$ and let $\mathbf{y}(t)=e^{-t}$ for all t. Clearly, we have $\mathbf{x}\succ\mathbf{y}$. Let $\mathbf{x}^n=\mathbf{x}\cdot\chi_{[0,n]}$ and let $\mathbf{y}^n=\mathbf{y}\cdot\chi_{[0,n]}+\chi_{[n,n+1]}$ for all $n\in\mathbb{N}$. For each $n\in\mathbb{N}$, and for each $t'\geq n+1$, we have

$$\int_{[0,t']}\mathrm{x}^n(t)d\mu(t)=2(1-e^{-n})<(1-e^{-n})+1=\int_{[0,t']}\mathrm{y}^n(t)d\mu(t)$$

Thus, we have $\mathbf{y}^n\succ\mathbf{x}^n$ for all n. By Lemma 1, $\{\mathbf{x}^n\}$ and $\{\mathbf{y}^n\}$ converge to $\mathbf{x}$ and $\mathbf{y}$, respectively, in the Mackey topology. Hence $\succ$ violates Mackey continuity. It is easy to show that $\succ$ satisfies the other axioms.

5. Concluding remarks

We conclude the paper by mentioning an open problem. The following statement is true for

l_∞ : Let τ be a Hausdorff, locally convex, linear topology strictly larger than the Mackey topology but strictly smaller than the sup-norm topology. Then one can find a social preference relation satisfying non-triviality, asymmetry, τ-continuity, and equity. In view of this result, we can say the Mackey topology is a maximal (in the sense of set-inclusion),Hausdorff, locally convex, linear topology leading to the impossibility type of results discussed in this paper. Whether a similar result holds for $\mathscr{L}_\infty$ remains open.

Notes

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3) If we followed Kaneko (1981, 1984), then we should call our axiom anonymity instead. We chose to stick to the nomenclature in the foregoing literature.

4) We could incorporate nonnegativity constraints on consumption without any difficulty.

5) $\mathbf{x}\circ\pi$ stands for the set $\{x\circ\pi \mid x\in\mathbf{x}\}$ which is indeed an equivalence class since π is a measure-preserving isomorphism on T.

6) For an alternative formulation of equity and the logical relation between the two, see Shinotsuka (1998).

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