

On the Fair Allocations in the Two-sided Matching Problems

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ABSTRACT

Our purpose is to discuss the existence of fair allocations in two-sided matching problems. The existence of fair allocations is not guaranteed in general, however, it is a dense property in an appropriate topology. Several other topological properties are explored for the sets of problems with or without fair allocations. We give a necessary and sufficient condition for the existence of fair allocations in the separable utility case. The lattice structure and monotonicity property of the set of fair allocations are also examined.

1 . Introduction

The purpose of this paper is to discuss the existence of fair allocations in two-sided matching problems. A two-sided matching problem consists of two disjoint sets of agents who have preference orderings over the agents on the opposite side of each. Typical examples are the marriage problems considered by Gale and Shapley (1962), the labor market models studied by Crawford and Knoer (1981), Kelso and Crawford (1982) and others²⁾. Our model is different from the marriage problems in the sense that agents' utility functions depend not only on their partners but also monetary transfers that they receive. It also differs from the labor market models in the sense that monetary transfers may not be pairwise. An allocation is a pair of an assignment of agents on one side to those on the other and a vector of monetary transfers. As usual, an allocations is fair if it is envy-free and efficient.

In recent years, a lot of authors have studied markets with large indivisibilities, containing one divisible good (called money) and possibly many indivisible goods which are consumed by each agent in at most one unit. In those models, agents choose objects but objects make no decision. On the other hand, in our setting, the both sides of the market are active and treated symmetrically.

In the literature on one-sided markets, many people concerned with the existence of fair

allocations and of competitive equilibria. The first theme was studied by Alkan, Demange, and Gale (1988), Crawford and Heller (1979), Maskin (1987), Svensson (1983), (1984), and (1988). The second one was studied by Gale (1984), Kaneko (1982), Quinzii (1984), Svensson (1983), and (1984).

In one-sided markets, fair allocations exist under very mild conditions and the set of fair allocations has the lattice structure. It also satisfies monotonicity property which means that if we start from a fair allocation and increase the total amount of money, then we may find a fair allocation which makes everyone better off. In two-sided markets, the conclusions drastically change. The existence of fair allocations is not ensured under very restricted assumptions. The lattice structure and monotonicity may fail to hold. These are bad news, however, we have some good news. First, the set of problems which have fair allocations is dense in an appropriate topology. Second, although the set is not open, it has a non-empty interior. Then, we may conclude that the existence of fair allocations is almost likely. Third, in the case of separable utility functions, the monotonicity property is satisfied. Finally, in the separable case, we obtain the necessary and sufficient condition for the existence of envy-free allocations.

In the next section, we explain the model and give an example of non-existence of fair allocations. We also clarify the topological structure of the sets of problems with or without fair allocations. In the third section, we examine the lattice property and monotonicity. In the fourth section, we consider the separable utility case. The fifth section gathers the proofs. The final section gives some remarks and an open question.

2. The Model and Examples

Let M and W denote the two disjoint finite sets of men and of women respectively. Assume that $|M|=|W|=n<\infty$. Each $m\in M$ has a utility function $u_m:W\times\mathbb{R}\rightarrow\mathbb{R}$. Similarly, each woman w has a utility function $u_w:M\times\mathbb{R}\rightarrow\mathbb{R}$. The total amount of money in the society is denoted by \hat{z} . A list $P=\{(u_a)_{a\in M\cup W}, \hat{z}\}$ is called a *two-sided matching problem*. We assume that for each $m\in M$ and each $w\in W$, u_m and u_w satisfy,

$$\lim_{z\rightarrow\infty} u_m(w, z) = \lim_{z\rightarrow\infty} u_w(m, z) = +\infty.$$

Furthermore, it is assumed that $u_m(w, z)$ and $u_w(m, z)$ are strictly increasing and continuous in z for all m and w . For a given problem P , an *allocation* is a pair (σ, z) of a bijection σ from M into W and a $2\times n$ dimensional vector $z=(z_a)_{a\in M\cup W}$. An allocation (σ, z) is *feasible* if $\sum_{a\in M\cup W} z_a = \hat{z}$. The bijection σ indicates the partner of each man. It is called an assignment. The vector

$z=(z_a)_{a \in M \cup W}$ describes a distribution of money. Note that z_a may be positive or negative. The set of all feasible allocations of P is denoted by $Z(P)$.

Definition 1. For a given problem P , allocation (σ, z) is *envy-free* if for all $m \in M$ and $w \in W$,

$$u_m(\sigma(m), z_{\sigma(m)}) \geq u_m(w, z_w) \text{ and } u_w(\sigma^{-1}(w), z_{\sigma^{-1}(w)}) \geq u_w(m, z_m).$$

Let $N(P)$ be the set of all envy-free allocations of P .

Definition 2. For a given problem P , allocation (σ, z) is *fair* if it is envy-free and is feasible. Let $F(P)$ denote the set of all fair allocations of P .

Definition 3. For a given problem P , allocation (σ', z') *Pareto dominates* allocation (σ, z) if for all $m \in M$ and $w \in W$,

$$u_m(\sigma'(m), z'_{\sigma'(m)}) \geq u_m(\sigma(m), z_{\sigma(m)})$$

and

$$u_w(\sigma'^{-1}(w), z'_{\sigma'^{-1}(w)}) \geq u_w(\sigma^{-1}(w), z_{\sigma^{-1}(w)})$$

and at least one of these inequalities is strict. Allocation (σ, z) is *Pareto optimal* if it is feasible and it is not Pareto dominated by any other feasible allocations. Let $PO(P)$ be the set of all Pareto optimal allocations.

In economies in which all goods are perfectly divisible, a feasible and envy-free allocation may not be Pareto optimal. In contrast, for the one-sided problems which involve finite number of indivisible goods, every feasible envy-free allocation is Pareto optimal. (See, Svensson (1983), Alkan, Demange, and Gale (1988).) In our case, this remains to be true.

Proposition 1. For any problem P , $F(P) \subseteq PO(P)$.

Definition 4. Let \mathcal{P} be the set of all two-sided matching problems. We endow \mathcal{P} with the product of the topology of uniform convergence on compact sets³⁾ and the Euclidean topology on \mathbb{R} . Define,

$$\mathcal{P}_+ = \{P \in \mathcal{P} | F(P) \neq \emptyset\} \text{ and } \mathcal{P}_\emptyset = \mathcal{P} \setminus \mathcal{P}_+^{4)}$$

Example 1. (Non-existence of envy-free allocations). Let $M = \{m_1, m_2\}$ and $W = \{w_1, w_2\}$.

Consider utility functions satisfying the following inequalities for any z .

$$\begin{aligned} u_{m_1}(w_1, z) &> u_{m_1}(w_2, z) \quad \text{and} \quad u_{m_2}(w_2, z) > u_{m_2}(w_1, z), \\ u_{w_1}(m_2, z) &> u_{w_1}(m_1, z) \quad \text{and} \quad u_{w_2}(m_1, z) > u_{w_2}(m_2, z). \end{aligned}$$

For instance, letting $\alpha > 0$, if we set

$$\begin{aligned} u_{m_1}(w_1, z) &= u_{m_2}(w_2, z) = z + \alpha \quad \text{and} \quad u_{m_1}(w_2, z) = u_{m_2}(w_1, z) = z, \\ u_{w_1}(m_2, z) &= u_{w_2}(m_1, z) = z + \alpha \quad \text{and} \quad u_{w_1}(m_1, z) = u_{w_2}(m_2, z) = z \end{aligned}$$

for any z , then the above inequalities are satisfied. Let $\hat{z} = 0$. Then, for the problem P given by the above data, $N(P) = \emptyset$. Indeed, consider the assignment $\sigma_1 = \{(m_1, w_1), (m_2, w_2)\}$ ⁵. Suppose that an allocation (σ_1, z) is envy-free. Then, $u_{w_1}(m_1, z_{m_1}) \geq u_{w_2}(m_2, z_{m_2}) > u_{w_1}(m_1, z_{m_2})$. Since u_{w_1} is monotonic in z , $z_{m_1} > z_{m_2}$. On the other hand, $u_{w_2}(m_2, z_{m_2}) \geq u_{w_2}(m_1, z_{m_1}) > u_{w_2}(m_2, z_{m_1})$. By the monotonicity, $z_{m_2} > z_{m_1}$, which is a contradiction. By applying the same argument to m_1 and m_2 , any allocation (σ_2, z) where $\sigma_2 = \{(m_1, w_2), (m_2, w_1)\}$ is not envy-free. Therefore, $N(P) = \emptyset$ and hence $F(P) = \emptyset$.

As this example shows, envy-free allocations may fail to exist. This contrasts to the one-sided problems in which fair allocations always exist under the same assumptions as ours. The non-existence is due to the fact that the model has two sides and the agents on the both sides are completely symmetric. Although the existence is not guaranteed in general, we can prove that the problems of having fair allocations constitutes a dense set.

Theorem 1. *The set \mathcal{P}_+ is dense in \mathcal{P} .*

Other topological structures of \mathcal{P}_+ and \mathcal{P}_\emptyset are summarized as follows.

Proposition 2. *The set \mathcal{P}_+ is not open.*

Theorem 2. *The set \mathcal{P}_+ has a non-empty interior.*

Theorem 1 says that non-existence is not robust. By Proposition 2, the existence may not be stable. By Theorem 2, the set \mathcal{P}_+ has non-empty interior and hence the existence is likely.

3. The Lattice Property and Monotonicity

For the one-sided problems, the set of envy-free allocations has the lattice property. Also, if the total amount of money increases, then we can always find a fair allocation which makes everyone better off. (See, Alkan-Demange-Gale (1988)). In this section, we show that the analogous lattice property and the nice comparative static result do not extend to the two-sided problems.

For a moment, let (σ, z) and (σ', z') be two fair allocations in a one-sided problem. Here, σ and σ' are assignments of indivisible items to agents and z and z' are n -dimensional vectors of monetary transfer. Alkan, Demange, and Gale (1988) define the lattice operation in the following way.

$$(\sigma \vee_M \sigma')(m) = \begin{cases} \sigma(m) & \text{if } u_m(\sigma(m), z_m) \geq u_m(\sigma'(m), z'_m). \\ \sigma'(m) & \text{otherwise.} \end{cases}$$

$$(z \vee_M z')_w = \max \{z_w, z'_w\}.$$

They prove that $(\sigma \vee_M \sigma', z \vee_M z')$ is also fair. Now, let P be a two-sided problem with $N(P) \neq \emptyset$. For any two allocations (σ, z) and (σ', z') in $F(P)$, it is quite natural to define $(\sigma \vee_M \sigma')(m)$ and $(z \vee_M z')_w$ in the same way as above. On the other hand, it is not obvious how to define $(z \vee_M z')_m$ for $m \in M$. But there are at least two possible ways. The first one is to define $(z \vee_M z')_m = \min \{z_m, z'_m\}$ and the second one is to define $(z \vee_M z')_m = \max \{z_m, z'_m\}$. We show that in each case the lattice property does not hold for the set $F(P)$ in general.

Example 2 (The Lattice Operation in the First Sense.)

Let P be the problem defined as follows.

$$M = \{m_1, m_2, m_3, m_4\}, \quad W = \{w_1, w_2, w_3, w_4\}, \quad \hat{z} = 6.$$

$$\begin{aligned} u_{m_1}(w_1, z) &= u_{m_2}(w_2, z) = z, \quad u_{m_1}(w_2, z) = u_{m_2}(w_1, z) = 5/3 + z/3, \\ u_{m_1}(w_3, z) &= u_{m_2}(w_4, z) = u_{m_2}(w_3, z) = u_{m_2}(w_4, z) = z/2, \\ u_{m_3}(w_1, z) &= u_{m_3}(w_2, z) = u_{m_4}(w_1, z) = u_{m_4}(w_2, z) = z/2, \\ u_{m_3}(w_3, z) &= u_{m_4}(w_4, z) = 2 + z/2, \quad u_{m_3}(w_4, z) = u_{m_4}(w_3, z) = 1 + 2z. \end{aligned}$$

For each $k=1, \dots, 4$, u_{w_k} is obtained from u_{m_k} substituting m into w . Consider assignments,

$$\begin{aligned} \sigma_1 &= \{(m_1, w_1), (m_2, w_2), (m_3, w_3), (m_4, w_4)\}, \\ \sigma_2 &= \{(m_1, w_2), (m_2, w_1), (m_3, w_4), (m_4, w_3)\}. \end{aligned}$$

and vectors,

$$\begin{aligned} z^1 &= ((z_{m_k})_{k=1}^4, (z_{w_k})_{k=1}^4) = ((3, 3, 0, 0), (3, 3, 0, 0)), \\ z^2 &= ((z_{m_k})_{k=1}^4, (z_{w_k})_{k=1}^4) = ((2, 2, 1, 1), (2, 2, 1, 1)). \end{aligned}$$

It is not difficult to see that $(\sigma_1, z^1), (\sigma_2, z^2) \in N(P)$ and $\sigma_1 \vee_M \sigma_2 = \sigma_3 = \{(m_1, w_1), (m_2, w_2), (m_3, w_4), (m_4, w_3)\}$. If we define $(z^1 \vee_M z^2)_m = \min \{z_m^1, z_m^2\}$, then $(\sigma_1 \vee_M \sigma_2, z^1 \vee_M z^2) = (\sigma_3, (3, 3, 2, 2), (2, 2, 0, 0))$. In this allocation, w_1 has envy.

Example 2'. (The Lattice Operation in the Second Sense.)

Let M, W, \hat{z} , and u_m 's are exactly same as in Example 2. Define,

$$\begin{aligned} u_{w_1}(m_1, z) &= u_{w_2}(m_2, z) = 2 + z/2, \quad u_{w_1}(m_2, z) = u_{w_2}(m_1, z) = 1 + 2z, \\ u_{w_1}(m_3, z) &= u_{w_1}(m_4, z) = u_{w_2}(m_3, z) = u_{w_2}(m_4, z) = z/2, \\ u_{w_3}(m_1, z) &= u_{w_3}(m_2, z) = u_{w_4}(m_1, z) = u_{w_4}(m_2, z) = z/2, \\ u_{w_3}(m_3, z) &= u_{w_4}(m_4, z) = z, \quad u_{w_3}(m_4, z) = u_{w_4}(m_3, z) = 5/3 + z/3. \end{aligned}$$

Let σ_1 and σ_2 be the same as in Example 2 and let

$$\begin{aligned} z^1 &= ((3, 3, 0, 0), (0, 0, 3, 3)), \\ z^2 &= ((2, 2, 1, 1), (1, 1, 2, 2)). \end{aligned}$$

If we define $(z^1 \vee_M z^2)_m = \max \{z_m^1, z_m^2\}$, then $(\sigma_1 \vee_M \sigma_2, z^1 \vee_M z^2) = (\sigma_3, (3, 3, 2, 2), (1, 1, 3, 3))$. In this allocation, w_1 has envy.

Alkan, Demange, and Gale (1988) show that if we start from a fair allocation in a one-sided problem and increase the total amount of money, then we can find a fair allocation which makes everyone strictly better off. This may not be true for a two-sided problem.

Example 3. Let $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$. Define P as follows.

$$\begin{aligned} u_{m_1}(w_1, z) &= u_{m_2}(w_1, z) = z, \quad u_{m_1}(w_2, z) = u_{m_3}(w_2, z) = 2 + z/2, \\ u_{m_1}(w_3, z) &= u_{m_2}(w_3, z) = u_{m_3}(w_1, z) = z/2, \\ u_{m_2}(w_2, z) &= u_{m_3}(w_3, z) = 2 + z, \\ u_{w_2}(m_1, z) &= u_{w_2}(m_1, z) = z, \\ u_{w_1}(m_2, z) &= u_{w_2}(m_2, z) = u_{w_3}(m_3, z) = 2 + z/2, \\ u_{w_1}(m_3, z) &= u_{w_3}(m_1, z) = z/2, \quad u_{w_2}(m_3, z) = u_{w_3}(m_2, z) = 2 + z. \end{aligned}$$

Let $\hat{z}=4$. Consider assignment $\sigma_2=\{(m_1, w_2), (m_2, w_1), (m_3, w_3)\}$. Then, allocation $(\sigma_2, z)\equiv(\sigma_2, (2, 0, 0), (2, 0, 0))$ is fair. Although the computation is rather tedious, it is not difficult to check that no envy free allocation gives each agent utility value strictly greater than 4 regardless of the total amount of money. Therefore, we can find no fair allocation which makes all agents strictly better off even if \hat{z} is increased.

4. The Class of Separable Preferences

In this section, we restrict our attention to the special class of problems in which every utility function is separable in money. In words, the marginal utility of money is independent of which person is your partner. This assumption is reasonable in many applications.

Definition 5. Utility function u_m is *separable in money* if $u_m(w, z) \geq u_m(w', z')$ implies $u_m(w, z + \delta) \geq u_m(w', z' + \delta)$ for all w, w', z, z', δ . The separability of utility function u_w is analogously defined. A *separable problem* P^s is a two-sided matching problem in which all utility functions are separable. Let \mathcal{P}^s be the set of all separable problems.

Note that utility function u_m (or u_w) is separable if and only if for some function f_m (or f_w), $u_m(w, z) = \alpha_{mw} + f_m(z)$ for all w and z , where $\alpha_{mw} = u_m(w, 0)$. (or $u_w(w, z) = \beta_{wm} + f_w(z)$ for all m and z , where $\beta_{wm} = u_w(m, 0)$.)

Suppose that utility function u_m is separable. Then, by a monotonic transformation, we obtain an equivalent representation still denoted by u_m which is of the form $u_m(w, z) = \alpha_{mw} + z$, where $\alpha_{mw} \geq 0$. Similarly, for a separable u_w , we have an equivalent representation $u_w(m, z) = \beta_{wm} + z$, where $\beta_{wm} \geq 0$. For each separable problem $P^s = \{(u_a)_{a \in M \cup W}, \hat{z}\}$, let $\{(\alpha_{mw}), (\beta_{wm}), \hat{z}\}$ be such an equivalent representation of P^s . Let v_α and v_β be the $n \times n$ matrices whose entries are α_{mw} and β_{wm} , respectively. Note that v_α and v_β can be viewed as assignment games considered by Shapley and Shubik (1972). Let x be a payoff vector $(x_a)_{a \in M \cup W}$. A pair (σ, x) of an assignment and a payoff vector is a stable outcome for assignment game v if and only if,

$$\begin{aligned} x_m + x_{\sigma(m)} &= v(m, \sigma(m)) \text{ for all } m \in M, \\ x_m + x_w &\geq v(m, w) \text{ for all } m \in M \text{ and } w \in W, \\ x_a &\geq 0 \text{ for all } a \in M \cup W. \end{aligned}$$

Let $C(v)$ be the set of all stable outcomes of v and let $AC(v) = \{(\sigma, x) \in C(v) \text{ for some } x\}$.

Now, we obtain a necessary and sufficient condition for the existence of fair allocations in the separable case.

Theorem 3. *A necessary and sufficient condition for a separable problem P^s to have a fair allocation is that for some equivalent representation $\{(\alpha_{mw}), (\beta_{wm})\}$ of the utility functions, $AC(v_\alpha) \cap AC(v_\beta) \neq \emptyset$.*

We also obtain the monotonicity property in this case.

Theorem 4. *Let $P^s = \{(u_a)_{a \in M \cup W}, \hat{z}\}$, be a separable problem and $(\sigma, z) \in F(P^s)$. Suppose that $\hat{z}' > \hat{z}$ and let $P^s = \{(u_a)_{a \in M \cup W}, \hat{z}'\}$. Then, there exists an allocation $(\sigma', z') \in F(P^s)$ which makes everyone strictly better off.*

5. Proofs

Lemma. *Let (σ, z) be an envy-free allocation of a problem P and let (τ, x) be any allocation and let $\tau(m) = w$. Then,*

$$\begin{aligned} u_m(\tau(m), x_{\tau(m)}) &> u_m(\sigma(m), z_{\sigma(m)}) \quad \text{implies} \quad x_w > z_w \\ u_m(\tau(m), x_{\tau(m)}) &\geq u_m(\sigma(m), z_{\sigma(m)}) \quad \text{implies} \quad x_w \geq z_w \\ u_m(m, x_m) &> u_m(\sigma^{-1}(w), z_{\sigma^{-1}(w)}) \quad \text{implies} \quad x_m > z_m \\ u_m(m, x_m) &\geq u_m(\sigma^{-1}(w), z_{\sigma^{-1}(w)}) \quad \text{implies} \quad x_m \geq z_m. \end{aligned}$$

Proof of the lemma. Because (σ, z) is envy-free, $u_m(\sigma(m), z_{\sigma(m)}) \geq u_m(w, z_w)$. By the assumption, $u_m(w, x_w) > u_m(\sigma(m), z_{\sigma(m)}) \geq u_m(w, z_w)$. Since u_m is increasing in money, $x_w > z_w$. The remaining implications are proved in the same way.

Proof of Proposition 1. Let (σ, z) be a fair allocation in problem P . Suppose that there is a Pareto dominating allocation (τ, x) . By the above lemma, $x_{\tau(m)} \geq z_{\tau(m)}$, $x_{\tau^{-1}(w)} \geq z_{\tau^{-1}(w)}$ for all $m \in M$ and $w \in W$ and at least one of them is a strict inequality. Then, $\sum_{a \in M \cup W} x_a > \sum_{a \in M \cup W} z_a = \hat{z}$. Hence, (τ, x) is not feasible.

Proof of Theorem 1. Let $P = \{(u_a)_{a \in M \cup W}, \hat{z}\}$ be an arbitrary problem. Let k be a positive integer. For each k , define

$$z_k^M = \max_{m \in M} \left\{ \max \left\{ u_m \left(w, \frac{\hat{z}}{2n} + k \right) \middle| w \in W \right\} \right\}.$$

Pick an assignment σ . Let x^k be a number such that $u_m(\sigma(m), \frac{\hat{z}}{2n} + x^k) > z_k^M$ for all $m \in M$. Since u_m is monotonic in money, such an x^k exists. For each $m \in M$, define,

$$\begin{aligned}
u_m^k(w, z) &= u_m(w, z) \quad \text{if } z \leq \frac{\hat{z}}{2n} + k, \\
u_m^k(w, z) &= u_m(w, \frac{\hat{z}}{2n} + k) \\
&\quad + \left[\frac{u_m(\sigma(m), \frac{\hat{z}}{2n} + k + x^k) - u_m(w, \frac{\hat{z}}{2n} + k)}{x^k} \right] \left\{ z - \left(\frac{\hat{z}}{2n} + k \right) \right\} \\
&\quad \text{if } z > \frac{\hat{z}}{2n} + k.
\end{aligned}$$

For each $w \in W$, if $u_m(m, \frac{\hat{z}}{2n} - k - x^k) \leq u_w(\sigma^{-1}(w), \frac{\hat{z}}{n} - k - x^k)$, then let $u_w^k(m, z) = u_w(m, z)$. Otherwise, define,

$$\begin{aligned}
u_m^k(m, z) &= u_w(m, z) \quad \text{if } z \geq \frac{\hat{z}}{2n} - k, \\
u_w^k(m, z) &= u_w(m, \frac{\hat{z}}{2n} - k) \\
&\quad + \left[\frac{u_w(m, \frac{\hat{z}}{2n} - k) - u_w(\sigma^{-1}(w), \frac{\hat{z}}{2n} - k - x^k)}{x^k} \right] \left\{ z - \left(\frac{\hat{z}}{2n} - k \right) \right\} \\
&\quad \text{if } z < \frac{\hat{z}}{2n} - k.
\end{aligned}$$

Let $P^k = \{(u_a^k)_{a \in M \cup W}, \hat{z}\}$. Because $\left[\frac{\hat{z}}{2n} - k, \frac{\hat{z}}{2n} + k \right] \rightarrow (-\infty, +\infty)$, the sequence $\{P^k\}$ converges to P . For each k , for each $m \in M$ and $w \in W$, let $z_m^k = \frac{\hat{z}}{2n} + k + x^k$ and $z_w^k = \frac{\hat{z}}{2n} - k - x^k$. Denote by z^k the vector $(z_a^k)_{a \in M \cup W}$. By construction, (σ, z^k) is a fair allocation in P^k . Therefore, $F(P^k)$ is non-empty. This proves the theorem.

Proof of Proposition 2. Let $M = \{m_1, m_2, \dots, m_n\}$, $W = \{w_1, w_2, \dots, w_n\}$. For all $m \in M$ and $w \in W$, let $u_m(w, z) = u_w(m, z) = 1 + z$. Put $\hat{z} = 0$. Then, the problem P defined above has fair allocations. For each positive integer k , define $u_{m_1}^k(w_1, z) = u_{m_2}^k(w_2, z) = \dots = u_{m_n}^k(w_n, z) = u_{w_1}^k(m_n, z) = u_{w_2}^k(m_{n-1}, z) = \dots = u_{w_n}^k(m_1, z) = 1 + z$ and for any other m and w , define $u_m^k(w, z) = u_w^k(m, z) = (1 - \frac{1}{k}) + z$. Let $P^k = \{(u_a^k)_{a \in M \cup W}, \hat{z}\}$. By the same argument as in Example 1, P^k has no fair allocation. Because P^k converges to P as $k \rightarrow \infty$, P is a boundary point of \mathcal{P}_+ .

Proof of Theorem 2. Define $u_{m_1}(w_1, z) = u_{m_2}(w_2, z) = \dots = u_{m_n}(w_n, z) = u_{w_1}(m_1, z) = u_{w_2}(m_2, z) = \dots = u_{w_n}(m_n, z) = 1 + z$. For any other m and w , let $u_m(w, z) = u_w(m, z) = z$. Let $\hat{z} = 0$. Then, this

problem P is in the interior of \mathcal{P}_+ . Indeed, if we put $\sigma = \{(m_1, w_1), (m_2, w_2), \dots, (m_n, w_n)\}$, then the allocation $(\sigma, 0) \in F(P)$. Because $u_m(\sigma(m), 0) = 1 > 0 = u_m(w, 0)$ and $u_m(\sigma^{-1}(m), 0) = 1 > 0 = u_m(m, 0)$ for all m and w , if u'_m and u'_w are close to u_m and u_w , respectively and if \hat{z}' is close to 0, then we have,

$$u'_w(\sigma(m), \frac{\hat{z}'}{2n}) > u'_m(w, \frac{\hat{z}'}{2n}) \quad \text{and} \quad u'_w(\sigma^{-1}(w), \frac{\hat{z}'}{2n}) > u'_m(m, \frac{\hat{z}'}{2n})$$

for all m and w . Therefore, any problems sufficiently close to P^k have a fair allocation.

Proof of Theorem 3. Suppose that $(\sigma, z) \in F(P^s)$. Let $\{(\alpha_{mw}), (\beta_{wm}), \hat{z}\}$ be an equivalent representation of P^s . Then, for all $m \in M$ and for all $w \in W$,

$$\alpha_{m\sigma(m)} + z_{\sigma(m)} \geq \alpha_{mw} + z_w \quad \text{and} \quad \beta_{w\sigma^{-1}(w)} + z_{\sigma^{-1}(w)} \geq \beta_{wm} + z_m.$$

By subtracting a sufficiently large number from each coordinate of z , we may think that all z 's in the above inequalities are non-positive. By adding a sufficiently large number to all α_{mw} 's and β_{wm} 's, we may assume that $\alpha_{m\sigma(m)} + z_{\sigma(m)} \geq 0$ and $\beta_{w\sigma^{-1}(w)} + z_{\sigma^{-1}(w)} \geq 0$. Let $x_m = \alpha_{m\sigma(m)} + z_{\sigma(m)}$ and $x_w = -z_w$. Denote by x the vector $(x_a)_{a \in M \cup W}$. Let $x'_m = -z_m$ and $x'_w = \beta_{w\sigma^{-1}(w)} + z_{\sigma^{-1}(w)}$. Denote by x' the vector $(x'_a)_{a \in M \cup W}$. Then, by definition,

$$x_m + x_{\sigma(m)} = \alpha_{m\sigma(m)} \quad \text{and} \quad x_m + x_w = \alpha_{m\sigma(m)} + z_{\sigma(m)} - z_w \geq \alpha_{mw}.$$

$$x'_m + x'_{\sigma(m)} = -z_m + \beta_{\sigma(m)m} + z_m = \beta_{\sigma(m)m}$$

$$\text{and} \quad x'_m + x'_w = -z_m + \beta_{w\sigma^{-1}(w)} + z_{\sigma^{-1}(w)} \geq \beta_{wm}.$$

It has been shown that $(\sigma, x) \in C(v_\alpha)$ and $(\sigma, x') \in C(v_\beta)$. Hence, $\sigma \in AC(v_\alpha) \cap AC(v_\beta)$. Conversely, suppose that for some equivalent representation $\{(\alpha_{mw}), (\beta_{wm}), \hat{z}\}$ there is an assignment σ such that $(\sigma, x) \in C(v_\alpha)$ and $(\sigma, x') \in C(v_\beta)$ for some x and x' . Let $z_w = -x_w$ and $z_m = -x'_m$. Then, for all m and w , $\alpha_{m\sigma(m)} + z_{\sigma(m)} = \alpha_{m\sigma(m)} - x_{\sigma(m)} = x_m \geq \alpha_{mw} - x_w = \alpha_{mw} + z_m$ and $\beta_{w\sigma^{-1}(w)} + z_{\sigma^{-1}(w)} = \beta_{w\sigma^{-1}(w)} - x'_{\sigma^{-1}(w)} = x'_w \geq \beta_{wm} - x'_m = \beta_{wm} + z_m$. Hence, if we denote the vector $(z_a)_{a \in M \cup W}$ by z , then the allocation (σ, z) is envy-free. Let $z'_a = z_a + \frac{1}{2n}(\hat{z} - \sum_{a \in M \cup W} z_a)$ for all $a \in M \cup W$ and $z' = (z'_a)_{a \in M \cup W}$. Then, the allocation (σ, z') is fair, so that $F(P^s) \neq \emptyset$.

Proof of Theorem 4. Let $(\sigma, z) \in F(P^s)$. Define $z'_a = z_a + \frac{1}{2n}(\hat{z}' - \hat{z})$. Then, (σ, z') is a fair allocation of the problem obtained from P^s by replacing \hat{z} with \hat{z}' . It is obvious that everyone is made strictly better off at (σ, z') .

6. Remarks and an Open Question

We should note that when proving the denseness of the set \mathcal{P}_+ , the approximating problems obtained in the proof of theorem 1 may not be separable. This suggests that the same argument is no longer valid for the set of separable problems. In fact, we cannot prove the ‘generic’ existence of fair allocations for the separable case.

Let \mathcal{P}_+^s be the set of separable problems which have a fair allocation and $\mathcal{P}_0^s = \mathcal{P}^s \setminus \mathcal{P}_0^s$. Then, we can prove the following.

Remark 1. The set \mathcal{P}_+^s may not be dense.

Proof. Let P^s be the separable problem given by the data;

$$\begin{aligned} u_{m_1}(w_1, z) &= u_{m_2}(w_2, z) = 1 + z, \quad u_{m_1}(w_2, z) = u_{m_2}(w_1, z) = z, \\ u_{w_1}(m_2, z) &= u_{m_2}(m_1, z) = 1 + z, \quad u_{w_1}(m_1, z) = u_{m_2}(m_2, z) = z. \\ \hat{z} &= 0. \end{aligned}$$

Since the inequalities in Example 1 of Section 2 are satisfied, $N(P^s) = \emptyset$. For any separable problem $P^{s'} = \{(u'_{m_i})_{i=1,2}, (u'_{w_i})_{i=1,2}, \hat{z}'\}$ which is sufficiently close to P^s , we have

$$\begin{aligned} u'_{m_1}(w_1, 0) &> u'_{m_1}(w_2, 0), \quad u'_{m_2}(w_2, 0) > u'_{m_2}(w_1, 0), \\ u'_{w_1}(m_2, 0) &> u'_{w_1}(m_1, 0), \quad u'_{w_2}(m_1, 0) > u'_{w_2}(m_2, 0), \end{aligned}$$

Suppose that an allocation (σ_1, z) is envy-free in $P^{s'}$. Then,

$$\beta_{w_1 m_1} + f_{w_1}(z_{m_1}) = u'_{w_1}(m_1, z_{m_1}) \geq u'_{w_1}(m_2, z_{m_2}) = \beta_{w_1 m_2} + f_{w_1}(z_{m_2}).$$

Hence, $f_{w_1}(z_{m_1}) - f_{w_1}(z_{m_2}) \geq \beta_{w_1 m_2} - \beta_{w_1 m_1} = \beta_{w_1 m_2} - f_{w_1}(0) - (\beta_{w_1 m_1} - f_{w_1}(0)) = u'_{w_1}(m_2, 0) - u'_{w_1}(m_1, 0) > 0$. Since f_{w_1} is strictly increasing, $z_{m_1} > z_{m_2}$.

On the other hand,

$$\beta_{w_2 m_2} + f_{w_2}(z_{m_2}) = u'_{w_2}(m_2, z_{m_2}) \geq u'_{w_2}(m_1, z_{m_1}) = \beta_{w_2 m_1} + f_{w_2}(z_{m_1}),$$

hence, $f_{w_2}(z_{m_2}) - f_{w_2}(z_{m_1}) \geq u'_{w_2}(m_1, 0) - u'_{w_2}(m_2, 0) > 0$. Since f_{w_2} is also strictly increasing, $z_{m_2} > z_{m_1}$, a contradiction. Applying the same argument to m_1 and m_2 , we can show that any allocation (σ_2, z) is not envy-free. Therefore, $N(P^s) = \emptyset$.

We already know that the set \mathcal{P}_+^s is not open in \mathcal{P}^s . The proof of Remark 1 indicates that for $n=2$, the set \mathcal{P}_0^s has non-empty interior. The following remark shows that the set \mathcal{P}_0^s has a boundary point.

Remark 2. In general, the set \mathcal{P}_0^s is not open. (Hence the set \mathcal{P}_+^s is not closed.)

Proof. Let $M=\{m_1, m_2\}$ and $W=\{w_1, w_2\}$. Define, $u_{m_1}(w_1, z)=u_{m_2}(w_2, z)=u_{w_1}(m_2, z)=1+z$, $u_m(w, z)=u_w(m, z)=z$ for all other m and w . Let $\hat{z}=0$ and P be the problem given by these data. It is not difficult to show that $F(P)=\emptyset$. Now, we construct a sequence $\{P^k\}$ of problems as follows.

For each positive integer k , define a function $g^k: \mathbb{R} \rightarrow \mathbb{R}$ by,

$$g^k(z) = \begin{cases} z & \text{if } z \leq k \\ \frac{1}{k}z + k - 1 & \text{if } z > k \end{cases}$$

Define $u_{w_2}^k(m_1, z) = g^k(z) - \frac{1}{k}$, $u_{w_2}^k(m_2, z) = g^k(z)$, $u_m^k(w, z) = u_m(w, z)$, and $u_w^k(m, z) = u_w(m, z)$ for any other m and w . Let $\hat{z}^k = \hat{z} = 0$ and P^k be the problem given by these data. It is easy to see that each P^k is a separable problem and the sequence $\{P^k\}$ approximates the original P . Let $\sigma_1 = \{(m_1, w_1), (m_2, w_2)\}$. For each k , consider allocation (σ_1, z^k) given by,

$$z^k = (z_{m_1}^k, z_{m_2}^k, z_{w_1}^k, z_{w_2}^k) = (k+1, k, -(k+\frac{1}{2}), -(k+\frac{1}{2})).$$

Since $u_{m_1}^k(w_1, z_{w_1}^k) = u_{m_2}^k(w_2, z_{w_2}^k) = 1 - (k + \frac{1}{2}) > -(k + \frac{1}{2}) = u_{m_1}^k(w_2, z_{w_2}^k) = u_{m_2}^k(w_1, z_{w_1}^k)$, no man has envy. Since $u_{w_1}^k(m_1, z_{m_1}^k) = k+1 = u_{w_2}^k(m_2, z_{m_2}^k)$, $u_{w_2}^k(m_2, z_{m_2}^k) = g^k(k) = k$, and $u_{w_1}^k(m_1, z_{m_1}^k) = g^k(k+1) - \frac{1}{k} = k$, no woman has envy. Because (σ_1, z^k) is feasible, $(\sigma_1, z^k) \in F(P^k)$. This shows that $P \in \mathcal{P}_0^s$ is contained in the boundary of \mathcal{P}_0^s .

Finally, we give an open problem. The difficulty of employing no-envy as fairness in our setting is, as indicated in Section 2, the non-existence. And even when it exists, the set of fair allocations can be large. In the one-sided problems, Tadenuma and Thomson (1989) propose refinements of the fair allocations and give an axiomatic evaluation of the refinements. The same line of research should be carried out in our setting.

Notes

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- 2) A comprehensive account on the literature of two-sided matching problems is given by Roth and Sotomayor (1990).
- 3) A sequence $\{u^t\}$ of functions on \mathbb{R} converges to u in this topology if and only if for any compact $K \subset \mathbb{R}$, $\sup_{z \in K} |u^t(z) - u(z)| \rightarrow 0$ as $t \rightarrow +\infty$. We write this as $u^t \rightarrow u$. A sequence $\{(u_a^t)_{a \in M \cup W}, \hat{z}^t\}$ of problems converges to a problem $\{(u_a)_{a \in M \cup W}, \hat{z}\}$ if and only if for all $a \in M \cup W$, $u_a^t \rightarrow u_a$ and $\hat{z}^t \rightarrow \hat{z}$.
- 4) For any two sets A and B , $A \setminus B = \{a \in A \mid a \notin B\}$.
- 5) Let σ be an assignment. When we write $\sigma = \{(m, w), (m', w'), \dots\}$, (m, w) means that $\sigma(m) = w$.

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