研究ノート

A Note on Aggregate Production Efficiency under Increasing Returns

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Abstract

In this note we consider an economy with one output and many inputs where production technologies are expressed by production functions and show that, if each production function is superadditive, it is necessary and sufficient for monopoly to achieve aggregate production efficiency that the function of the maximum value of individual production functions at each input is superadditive.

1 Introduction

It is well known that in an economy with convex technologies, profit-maximizing behavior of firms leads to both individual and aggregate production efficiency[1]. In other words, given prices, if each firm operates at a profit-maximizing production plan, it is on the individual production frontier; and the aggregate production plan is on the frontier of the aggregate production set.

By contrast, under non-convex or increasing returns to scale technologies, aggregate production efficiency cannot be achieved through individual maximizing behavior[2]. Under these technologies monopoly production may be required for aggregate production efficiency. This is explained by the fact that increasing returns yield scale merits of production and favor monopoly over decentralized production where two or more firms (technologies) are simultaneously active. That is, under these technologies monopoly may produce more output than two or more firms combined can do.

Hamano (1996) considers an economy where there are many firms with increasing returns to scale technologies and attempts to derive sufficient conditions ensuring that monopoly achieves aggregate production efficiency. More specially, Hamano (1996) examines an economy with one output and many inputs where production technologies are expressed by production functions, and obtains such sufficient conditions which can be interpreted as non-decreasing "generalized" average productivity of inputs for each firm.

In this note, we examine an economy with increasing returns technologies and derive a necessary and sufficient condition for monopoly to achieve aggregate production efficiency. More precisely we consider an economy with one output and many inputs where production technologies are expressed by production functions which is the same setting as in Hamano (1996). We show that, if each production function is superadditive, it is necessary and sufficient for monopoly to achieve aggregate production efficiency that the function of maximum value of individual production functions at each input is superadditive[3)].

The organization of the paper is as follows: Section 2 presents a basic framework and a concept of an aggregate production function. In Section 3 we derive a necessary and sufficient condition that monopoly achieves production efficiency in an economy where production technologies are expressed by production sets.

2 The Model

Let us consider an economy with one output, m inputs and H firms where $H \geq 2$. The technology of the h-th firm ($h=1, 2, \cdots, H$) is represented by a production function f^h defined from R^m_+ to R_+. That is, given a vector of inputs $z=(z_1, z_2, \cdots, z_m) \in R^m_+$, $f^h(z)$ is a maximum amount of output. For each h, a production function f^h is assumed to be non-decreasing and to satisfy the condition $f^h(0)=0$.

We recall the definition of a superadditive function.

Definition 1 *A function $f: R^m_+ \to R_+$ is called superadditive if, for all z and z',*

$$f(z+z') \geq f(z)+f(z').$$

We next present the definition of the aggregate production function at each input $z \in R^m_+$, given individual production functions.

Definition 2 *Given individual production function f^h ($h=1, 2, \cdots, H$), the aggregate production function $F: R^m_+ \to R_+$ is defined as*

$$F(z) \equiv \max_{z=\Sigma_{h=1}^H z^h,\, z^h \in R^m_+} \sum_{h=1}^{H} f^h(z^h). \tag{1}$$

We also define a function of the maximum value of the individual output at each input $z \in R^m_+$.

Definition 3 *Given individual production function* f^h $(h=1, 2, \cdots, H)$, *the function of the maximum value of the individual output,* G: $R_+^m \to R_+$, *is defined as*

$$G(z) \equiv \max_{h=1, 2, \cdots, H} f^h(z). \tag{2}$$

Note that, if $G(z)=F(z)$ for all $z \in R_+^m$, then the monopoly achieves aggregate production efficiency.

3 The Result

Our result is that the monopoly achieves aggregate production efficiency if and only if the function G is superadditive.

Theorem 1 *Suppose that* f^h *is superadditive for all* h. *Then, the function* $F(z)$ *is equal to* $G(z)$ *for all* $z \in R_+^m$ *if and only if* G *is superadditive.*

Proof.

Let us first show that, if $F(z)=G(z)$ for all $z \in R_+^m$, then the function $G(z)$ is superadditive. Fix $z', z'' \in R_+^m$. It suffices to show that

$$G(z'+z'') \geq G(z')+G(z''). \tag{3}$$

We now pick up indices of firms h', h'' associated with z', z'', respectively, as follows:

$$G(z') = f^{h'}(z'), \tag{4}$$
$$G(z'') = f^{h''}(z''). \tag{5}$$

When we choose $\{z^h\}_{h=1}^H$ satisfying $z'+z''=\sum_{h=1}^H z^h$, it follows from the equality of $F(z)$ with $G(z)$ that

$$G(z'+z'') = F(z'+z'') \geq \sum_{h=1}^H f^h(z^h). \tag{6}$$

If $h' \neq h''$, then we have

$$F(z'+z'') \geq f^{h'}(z')+f^{h''}(z''). \tag{7}$$

Moreover, if $h'=h''$, then it follows from superadditivity of f^h that

$$F(z'+z'') \geq f^{h'}(z'+z'') \geq f^{h'}(z')+f^{h'}(z'') = f^{h'}(z')+f^{h''}(z''). \tag{8}$$

In either case, we have

$$G(z'+z'') \geq f^{h'}(z')+f^{h''}(z''). \tag{9}$$

Therefore, it follows from (4), (5) and (6) that

$$G(z'+z'') \geq G(z')+G(z''), \tag{10}$$

which is to be shown.

Conversely, suppose that $G(z)$ is superadditive. We need to show that $F(z)=G(z)$ for all $z \in R_+^m$. Note from the definition of F that $F(z) \geq G(z)$. Suppose on the contrary that there exists $\hat{z} \in R_+^m$ such that $F(\hat{z})>G(\hat{z})$. If we set $F(\hat{z})=\sum_{h=1}^{H} f^h(\hat{z}^h)$ where $\hat{z}=\sum_{h=1}^{H} \hat{z}^h$, then it follows from superadditivity of G that

$$\sum_{h=1}^{H} f^h(\hat{z}^h) = F(\hat{z}) > G(\hat{z}) \geq \sum_{h=1}^{H} G(\hat{z}^h). \tag{11}$$

Now, by the definition of G, we have the relationship

$$G(\hat{z}^h) \geq f^h(\hat{z}^h) \quad \text{for all } h, \tag{12}$$

and, therefore,

$$\sum_{h=1}^{H} G(\hat{z}^h) \geq \sum_{h=1}^{H} f^h(\hat{z}^h). \tag{13}$$

Thus, we combine inequalities (11) and (13) to yield the relationship

$$\sum_{h=1}^{H} f^h(\hat{z}^h) > \sum_{h=1}^{H} f^h(\hat{z}^h), \tag{14}$$

which is a contradiction. **Q.E.D.**

Notes

1) See Mas-Colell et al. (1995, pp. 147–149) for example.
2) See Hamano (2011) for details.
3) Without superadditivity of individual production functions, Ginsberg (1974) considered a nicely convex-concave production function; and derived a condition on aggregate production efficiency in a case of one input and one output.

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References

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