Aggregation of generalized Cobb-Douglas production functions as a solution of non-convex optimization

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Abstract

We consider an economy with a special class of two generalized Cobb-Douglas production functions exhibiting increasing returns and provide a necessary and sufficient condition for monopoly to achieve aggregate production efficiency. We prove this result by finding a local maximum of non-convex optimization problem and check whether it is a global maximum or not. As corollaries, we obtain sufficient conditions for efficiency of monopoly production in a more general class.

1 Introduction

Convexity plays an important role in economic analysis. Under convexity of production and preference, price-taking behavior leads to Pareto efficient allocation. This implies that all of individual production plans are on the boundary of individual production sets and an aggregate production plan is on the boundary of aggregate production set; that is, decentralized production results in aggregate production efficiency[1)]. However, without convexity of production sets, decentralized production where two or more firms (technologies) are simultaneously active may not yield aggregate production efficiency[2)]. This is due to a scale merit arising from the nature of increasing returns. Instead, monopoly production where all other firms are inactive is more efficient than decentralized one.

Interestingly, increasing returns does not always imply aggregate production efficiency by a monopoly. When the degree of increasing returns to scale is mild, two or more firms combined may produce more output than monopoly. So, it is interesting to know the condition regarding the degree of increasing returns to ensure monopoly to achieve aggregate production efficiency[3)]. In this paper we consider an economy with one output, two inputs and two increasing returns technologies expressed by "generalized" Cobb-

Douglas production functions and derive sufficient conditions for monopoly to achieve aggregate production efficiency under increasing returns.

Similar sufficient conditions are obtained by Hamano (1996, 2011a, b). Hamano (1996) derived a sufficient condition for monopoly to be efficient in an economy with one output and many inputs, which can be interpreted as non-decreasing "generalized" average productivity of inputs for each technology. He also obtained as a corollary a special class of generalized Cobb-Douglas production functions that leads to aggregate production efficiency by a monopoly. Hamano (2011a, b) considered an economy with one output and two inputs where there are two production technologies expressed by generalized Cobb-Douglas production functions and derived alternative sufficient conditions for efficiency of monopoly production.

Let us now turn to the technical aspect of the problem. Examination of aggregate production efficiency is equivalent to finding an aggregate production function. If we look at the problem of deriving an aggregate production function under increasing returns as an optimization problem, it is a non-convex maximization problem and we cannot deal with this by using standard techniques of optimization under convexity. Note that from a view point of maximization, efficient monopoly production can be regarded as a "corner" solution while efficient production of two active firms can be regarded as an interior solution. To attack this non-convex optimization problem, Hamano (2011a, b) examined the condition under which the second order condition of the local maximum is not satisfied at those points satisfying the first order condition and derived a sufficient condition ensuring corner solutions. Our approach is completely different. We first consider a special class of two "symmetric" generalized Cobb-Douglas production functions under increasing returns and provide a necessary and sufficient condition for monopoly to achieve aggregate production efficiency. We prove this result by finding a local maximum of non-convex optimization problem and check whether it is a global maximum. As corollaries, we also obtain sufficient conditions for efficiency of monopoly production in a general class.

The organization of the paper is as follows: Section 2 presents a basic framework and a concept of an aggregate production function. Results in Hamano's (1996, 2011a, b) are also explained. In Section 3 we present main results and their proofs. Section 4 provides proofs of Lemmata that are used in the proofs of main results in Section 3. In Section 5 we provide an illustrating example to compare our results with Hamano (2011a, b)'s. Finally, we make some remarks on further research in Section 6.

2　Model and Review

Let us consider an economy with one output, two inputs and two firms. The technology of the h-th firm ($h=1, 2$) is represented by a production function f^h defined from R^2_+ to R_+. That is, given a vector of inputs $(x, y) \in R^2_+$, $f^h(x, y)$ is a maximum amount of output. For each h, a production function f^h is assumed to be non-decreasing and to satisfy the condition $f^h(0)=0$.

We first present the definition of the aggregate production function at each input $(x, y) \in R^2_+$, given individual production functions.

Definition 1　*Given individual production functions f^1 and f^2, the aggregate production function $AF : R^2_+ \to R_+$ is defined as*

$$AF(x, y) \equiv \max_{\substack{(x, y)=(x_1, y_1)+(x_2, y_2) \\ (x^h, y^h) \in R^2_+ (h=1, 2)}} \{f^1(x_1, y_1)+f^2(x_2, y_2)\}, \qquad \textit{for each } (x, y) \in R^2_+. \quad (1)$$

Throughout the paper we assume that production functions are expressed by generalized Cobb-Douglas types.

Assumption 2　*Production function of firm h is expressed by*

$$f^1(x, y) = A_1 x^{\alpha_1} y^{\beta_1} \quad \textit{and} \quad f^2(x, y) = A_2 x^{\alpha_2} y^{\beta_2}$$

where $A_h, \alpha_h, \beta_h > 0$ and $\alpha_h + \beta_h \geq 1$ for $h=1, 2$.

We first present the following result due to Hamano (1996, Example 3), in which a sufficient condition for monopoly to achieve aggregate production efficiency is derived. Note that his result is proved in a general framework where production functions are of general form and both the number of inputs and that of firms may be more than two. The same result is also derived in Hamano (2011b) by examining the second order condition for the local maximum of $A_1 x^{\alpha_1} y^{\beta_1} + A_2(\bar{x}-x)^{\alpha_2}(\bar{y}-y)^{\beta_2}$ where $(\bar{x}, \bar{y})$ is a vector of available amounts of inputs. Throughout the paper we assume that $\bar{x}>0$ and $\bar{y}>0$.

Proposition 3 (Hamano, 1996, Example 3) *If* $\min\{\alpha_1, \alpha_2\}+\min\{\beta_1, \beta_2\}\geq 1$ *is satisfied, then we obtain the inequality, for all* $(x, y)\in R^2_+$,

$$A_1x^{\alpha_1}y^{\beta_1}+A_2(\bar{x}-x)^{\alpha_2}(\bar{y}-y)^{\beta_2} \leq \max(A_1\bar{x}^{\alpha_1}\bar{y}^{\beta_1}, A_2\bar{x}^{\alpha_2}\bar{y}^{\beta_2}),$$

and

$$AF(x, y) = \max(A_1x^{\alpha_1}y^{\beta_1}, A_2x^{\alpha_2}y^{\beta_2}) \quad \textit{for all } (x, y)\in R^2_+.$$

The next result is proved in Hamano (2011a) where an alternative sufficient condition for monopoly to achieve aggregate production efficiency is derived by examining the second order condition for the local maximum.

Proposition 4 (Hamano, 2011a, Theorem 1) *Let us define* α *and* β *as follows:*

$$\alpha = \min(\alpha_1, \beta_2), \tag{2}$$

$$\beta = \min(\alpha_2, \beta_1). \tag{3}$$

If $\alpha+\beta<4\alpha\beta$, *the following inequality holds for all* $(x, y)\in R^2_+$ *with* $0\leq x\leq\bar{x}$ *and* $0\leq y\leq\bar{y}$,

$$A_1x^{\alpha_1}y^{\beta_1}+A_2(\bar{x}-x)^{\alpha_2}(\bar{y}-y)^{\beta_2} \leq \max(A_1\bar{x}^{\alpha_1}\bar{y}^{\beta_1}, A_2\bar{x}^{\alpha_2}\bar{y}^{\beta_2}), \tag{4}$$

and

$$AF(x, y) = \max(A_1x^{\alpha_1}y^{\beta_1}, A_2x^{\alpha_2}y^{\beta_2}) \quad \textit{for all } (x, y) \in R^2_+. \tag{5}$$

Note that Hamano (2011b) integrated Proposition 4 with Proposition 3. Thus, Hamano (2011b) requires weaker conditions than Hamano (1996). However, the latter is derived from the result in more general setting where there are more than two technologies. So, Hamano (1996)'s and (2011a)'s are not directly comparable.

In the next section, we derive sufficient conditions for efficiency of monopoly production, weaker than Proposition 4 in the economy with two technologies and two inputs.

3 Main Result

Let us now present main results of this paper. The first one is concerning the special case where $\bar{x}=\bar{y}=1$, $A_1=A_2=1$, $\alpha_1=\beta_2=\alpha$ and $\alpha_2=\beta_1=\beta$ which leads to $f^1(\bar{x}, \bar{y})=f^2(\bar{x}, \bar{y})=1$. The following result gives a necessary and sufficient condition for efficiency of monopoly production in this special case.

Theorem 5 *Let $\alpha, \beta > 0$ with $\alpha+\beta \geq 1$. A necessary and sufficient condition that $x^{\alpha}y^{\beta} + (1-x)^{\beta}(1-y)^{\alpha} \leq 1$ for all $(x, y) \in [0,1] \times [0,1]$ is that*

$$\frac{\alpha^{\alpha}\beta^{\beta}}{(\alpha+\beta)^{\alpha+\beta}} \leq \frac{1}{2}, \tag{6}$$

and

$$F(1,1) = \max_{\substack{0 \leq x_1 \leq 1 \\ 0 \leq y_1 \leq 1}} \{x_1^{\alpha}y_1^{\beta} + (1-x_1)^{\beta}(1-y_1)^{\alpha}\} = \begin{cases} 1 & \text{if } \dfrac{\alpha^{\alpha}\beta^{\beta}}{(\alpha+\beta)^{\alpha+\beta}} \leq \dfrac{1}{2}, \\ \dfrac{2\alpha^{\alpha}\beta^{\beta}}{(\alpha+\beta)^{\alpha+\beta}} & \text{otherwise.} \end{cases} \tag{7}$$

The next result relaxes the condition of Theorem 5 and covers more general cases. However, we can only give a sufficient condition for efficiency of monopoly production in Corollary 6.

Corollary 6 *Let $A_1, A_2 > 0$ and $\bar{x}, \bar{y} > 0$. Moreover, $\alpha, \beta > 0$ with $\alpha+\beta \geq 1$. Suppose that the following condition is satisfied:*

$$\frac{\alpha^{\alpha}\beta^{\beta}}{(\alpha+\beta)^{\alpha+\beta}} \leq \frac{1}{2}. \tag{8}$$

Then, we have, for all $(x, y) \in [0, \bar{x}] \times [0, \bar{y}]$,

$$A_1x^{\alpha}y^{\beta} + A_2(\bar{x}-x)^{\beta}(\bar{y}-y)^{\alpha} \leq \max(A_1\bar{x}^{\alpha}\bar{y}^{\beta}, A_2\bar{x}^{\beta}\bar{y}^{\alpha}), \tag{9}$$

and

$$AF(x, y) = \max(A_1x^{\alpha}y^{\beta}, A_2x^{\beta}y^{\alpha}) \quad \textit{for all } (x, y) \in R^2_+.$$

The results so far are all concerning the case of two symmetric generalized Cobb-Douglas functions, i.e., $\alpha_1 = \beta_2 = \alpha$ and $\alpha_2 = \beta_1 = \beta$. The last result gives a sufficient condition for efficiency of monopoly production in more general case where generalized Cobb-Douglas functions are not symmetric.

Corollary 7 Let $A_1, A_2 > 0$ and $\bar{x}, \bar{y} > 0$. Suppose that the following conditions are satisfied

$$\alpha = \min(\alpha_1, \beta_2), \tag{10}$$

and

$$\beta = \min(\alpha_2, \beta_1). \tag{11}$$

If $2\alpha^{\alpha}\beta^{\beta}/(\alpha+\beta)^{\alpha+\beta} \leq 1$ *is satisfied, then we have, for all* $(x, y) \in [0, \bar{x}] \times [0, \bar{y}]$,

$$A_1 x^{\alpha_1} y^{\beta_1} + A_2(\bar{x}-x)^{\alpha_2}(\bar{y}-y)^{\beta_2} \leq \max(A_1 \bar{x}^{\alpha_1}\bar{y}^{\beta_1}, A_2 \bar{x}^{\alpha_2}\bar{y}^{\beta_2}), \tag{12}$$

and

$$AF(x, y) = \max(A_1 x^{\alpha_1} y^{\beta_1}, A_2 x^{\alpha_2} y^{\beta_2}) \quad \textit{for all } (x, y) \in R^2_+.$$

We now start with the proof of Theorem 5.

Proof of Theorem 5

We first define a function $F : [0, 1] \times [0, 1] \to R_+$ as

$$F(x, y) = x^{\alpha} y^{\beta} + (1-x)^{\beta}(1-y)^{\alpha}.$$

Since $F(1, 1) = F(0, 0) = 1$, our goal is to derive a necessary and sufficient condition that, for all $(x, y) \in [0, 1] \times [0, 1]$,

$$F(x, y) \leq F(1, 1) = F(0, 0).$$

Let us now define the maximization problem (M) as follows:

$$\text{(M)} \qquad \max_{(x, y) \in [0, 1] \times [0, 1]} F(x, y).$$

Then, we look for the condition guaranteeing the corner solution for (M). For this goal we first examine when the function $F(x, y)$ achieves a local maximum in the interior of $[0, 1] \times [0, 1]$, and derive conditions guaranteeing that the maximum value is less than or equal to $F(1, 1) = F(0, 0) = 1$. A detailed outline of the proof is as follows:

Step 1: We first present some preliminary results (Lemma 8, 9, 10) which will be repeatedly used in the proof. Lemma 8 gives a characterization of the solution of the maximization problem (M) including the property that any (x, y) satisfying the first order condition may be written as $(x, l(x))$ where $l(x) = \beta^2 x/(\alpha^2 + (\beta^2 - \alpha^2)x)$. This allows us to focus exclusively upon the set of the combination $(x, l(x))$. Next, Lemma 9 asserts that examination of the first order condition for the maximum of $L(x) \equiv F(x, l(x))$ is equivalent to studying that of $F(x, y)$. Finally, Lemma 10 shows that the first order condition for (M) is always satisfied at

$(\tilde{x}, \tilde{y}) \equiv (\alpha/(\alpha+\beta), \beta/(\alpha+\beta))$. We also claim that the second order condition $F_{xx}(\tilde{x}, \tilde{y})<0$, $F_{yy}(\tilde{x}, \tilde{y})<0$ and $F_{xx}(\tilde{x}, \tilde{y})F_{yy}(\tilde{x}, \tilde{y})-(F_{xy}(\tilde{x}, \tilde{y}))^2>0$ of a local maximum is satisfied if $\alpha+\beta>4\alpha\beta$. That is, $F(x, y)$ achieves a local maximum at $(\tilde{x}, \tilde{y})$.

The remaining of the proof will be divided into three cases depending upon the sign of $\alpha+\beta-4\alpha\beta$.

Step 2: If $\alpha+\beta<4\alpha\beta$, then we shall claim that there is no local maximum in the interior of $[0,1]\times[0,1]$ by showing that the second order condition for a local maximum of $F(x, y)$ is violated at $(x, \ell(x))$. Therefore, in this case the maximization problem (M) has corner solutions; i.e., $F(x, y)<1=F(1,1)=F(0,0)$ for all $(x, y)\in(0,1)\times(0,1)$

Step 3: If $\alpha+\beta=4\alpha\beta$, then we shall show that $F(x, y)$ may possibly achieve a unique local maximum at $(\tilde{x}, \tilde{y}) \equiv (\alpha/(\alpha+\beta), \beta/(\alpha+\beta))$[4]. However, we shall also prove that $F(\tilde{x}, \tilde{y})\leq 1=F(1,1)=F(0,0)$, which also implies corner solutions.

Step 4: If $\alpha+\beta>4\alpha\beta$, we claim that $F(x, y)$ achieves a unique local maximum in the interior at $(\tilde{x}, \tilde{y})$. Since $F(\tilde{x}, \tilde{y})=2\alpha^\alpha\beta^\beta/(\alpha+\beta)^{\alpha+\beta}$, a necessary and sufficient condition that $F(x, y)=x^\alpha y^\beta+(1-x)^\beta(1-y)^\alpha\leq 1$ for all $(x, y)\in[0,1]\times[0,1]$ is

$$\frac{\alpha^\alpha\beta^\beta}{(\alpha+\beta)^{\alpha+\beta}} \leq \frac{1}{2}.$$

In other words, the maximization problem (M) has corner solutions if and only if $2\alpha^\alpha\beta^\beta/(\alpha+\beta)^{\alpha+\beta}\leq 1$. Note that according to results in Step 2 and 3, $\alpha+\beta\leq 4\alpha\beta$ implies $2\alpha^\alpha\beta^\beta/(\alpha+\beta)^{\alpha+\beta}\leq 1$.

To this end, we first show that, if there is a local maximum at some $(\hat{x}, \hat{y})\neq(\tilde{x}, \tilde{y})$, then $L(x)$ is minimized at some x^* between $\hat{x}$ and $\tilde{x}$; however, $F(x, y)$ is shown to have a local maximum at $(x^*, l(x^*))$ (Lemma 11); so is a local maximum for $L(x)$, which is a contradiction. Finally, we show that the condition $2\alpha^\alpha\beta^\beta/(\alpha+\beta)^{\alpha+\beta}\leq 1$ yields $F(\tilde{x}, \tilde{y})\leq F(1,1)=F(0,0)=1$, which implies corner solutions of the maximization problem (M).

Let us now proceed to the proof.

Step 1: We shall present a series of lemmata, which will be proved in Section 4. We first give a characterization of the solution of the maximization problem (M), which is summarized in Lemma 8.

Lemma 8 *Let a function* $l : [0,1] \to R$ *be defined by*

$$l(x) \equiv \frac{\beta^2 x}{\alpha^2 + (\beta^2 - \alpha^2)x},$$

and $SOC : [0,1] \to R$ by

$$SOC(x) \equiv -(\alpha-\beta)(\alpha^2-\beta^2)x^2 + (\alpha^2-\beta^2)(2\alpha-1)x + (1-\alpha-\beta)\alpha^2.$$

If $F(x, y)$ *satisfies the first order condition for a local maximum at* (x', y'), *then*

$$y' = l(x'), \tag{13}$$

$$F_{xx}(x', y') = \alpha(x')^{\alpha-2} l(x')^{\beta}(1-x')^{-1}\{(\alpha-1)-(\alpha-\beta)x'\}, \tag{14}$$

$$F_{yy}(x', y') = \beta(x')^{\alpha} l(x')^{\beta-2}(1-l(x'))^{-1}\{(\beta-1)+(\alpha-\beta)l(x')\}, \tag{15}$$

and

$$F_{xx}(x', y')F_{yy}(x', y') - (F_{xy}(x', y'))^2 = \frac{\alpha\beta(x')^{2\alpha-2} l(x')^{2\beta-2}}{(1-x')(1-l(x'))(\alpha^2-(\alpha^2-\beta^2)x')} SOC(x'). \tag{16}$$

This says that, in order to check whether the second order condition for the local maximum of $F(x, y)$ is satisfied or not at any (x, y) satisfying the first order condition, it suffices to examine the sign of $SOC(x)$. Since any (x, y) satisfying the first order condition may be written as $(x, l(x))$ where $l(x) = \beta^2 x / \{\alpha^2 + (\beta^2 - \alpha^2)x\}$, Lemma 8 allows us to focus upon the set of the combination $(x, l(x))$; i.e., the maximization problem (M) comes down to the maximization of the function of one variable x. Thus, we define $L : [0,1] \to R_+$ as

$$L(x) \equiv F(x, l(x)).$$

The next lemma asserts that examination of the first order condition for the maximum of $L(x) = F(x, l(x))$ is equivalent to studying that of (M).

Lemma 9 *Suppose that* $0 < x < 1$. *A necessary and sufficient condition for* $dL(x)/dx = 0$ *is* *that* $F_x(x, l(x)) = F_y(x, l(x)) = 0$; *i.e.*,

$$\frac{dL(x)}{dx} = 0 \iff F_x(x, l(x)) = F_y(x, l(x)) = 0. \tag{17}$$

Finally, the following lemma shows that $(\tilde{x}, \tilde{y}) \equiv (\alpha/(\alpha+\beta), \beta/(\alpha+\beta))$ always satisfies the first order condition for (M) and also satisfies the second order condition $F_{xx}(\tilde{x}, \tilde{y}) < 0$, $F_{yy}(\tilde{x}, \tilde{y}) < 0$ and $F_{xx}(\tilde{x}, \tilde{y}) F_{yy}(\tilde{x}, \tilde{y}) - (F_{xy}(\tilde{x}, \tilde{y}))^2 > 0$ under $\alpha+\beta > 4\alpha\beta$. That is, $F(x, y)$ achieves a local maximum at $(\tilde{x}, \tilde{y})$ if $\alpha+\beta > 4\alpha\beta$.

Lemma 10 *The first order condition for a local maximum of* $F(x, y)$ *is satisfied at* $(\tilde{x}, \tilde{y})$; *moreover, we have the following relations:*

$$F(\tilde{x}, \tilde{y}) = \frac{2\alpha^\alpha \beta^\beta}{(\alpha+\beta)^{\alpha+\beta}}, \tag{18}$$

$$F_{xx}(\tilde{x}, \tilde{y}) = F_{yy}(\tilde{x}, \tilde{y}) = \frac{\alpha^{\alpha-1}\beta^{\beta-1}}{(\alpha+\beta)^{\alpha+\beta-2}} \times (2\alpha\beta - \alpha - \beta), \tag{19}$$

$$SOC(\tilde{x}) = \frac{\alpha\beta}{\alpha+\beta}(\alpha+\beta-4\alpha\beta). \tag{20}$$

In addition, $(\tilde{x}, \tilde{y})$ *satisfies the second order condition for the local maximum of* $F(x, y)$ *if* $\alpha+\beta > 4\alpha\beta$.

Step 2: We shall show that, if $\alpha+\beta-4\alpha\beta < 0$, then there is no local maximum for $F(x, y)$ and we have the corner solutions. Although this result has been already proven in Hamano (2011a), we shall reproduce the proof because we also use the same argument in Step 3. For this goal it is sufficient to show that the second order condition for the local maximum of $F(x, y)$ is violated at any (x, y) satisfying the first order condition. Because of Lemma 8 it suffices to show that $SOC(x) < 0$ for all $x \in [0, 1]$. If we set $SOC(x) = 0$, i.e.,

$$-(\alpha-\beta)(\alpha^2-\beta^2)x^2 + (\alpha^2-\beta^2)(2\alpha-1)x + (1-\alpha-\beta)\alpha^2 = 0, \tag{21}$$

then we have a quadratic equation of x. Now, the discriminant D_1 of this quadratic equation (21) can be expressed as follows:

$$\begin{aligned} D_1 &= (\alpha-\beta)^2(\alpha+\beta)^2(2\alpha-1)^2 + 4(\alpha-\beta)(\alpha^2-\beta^2)(1-\alpha-\beta)\alpha^2 \\ &= (\alpha-\beta)^2(\alpha+\beta)\{(\alpha+\beta)(2\alpha-1)^2 + 4\alpha^2(1-\alpha-\beta)\} \\ &= (\alpha-\beta)^2(\alpha+\beta)(\alpha+\beta-4\alpha\beta). \end{aligned} \tag{22}$$

If $\alpha+\beta-4\alpha\beta < 0$ or $\alpha+\beta < 4\alpha\beta$, then $D_1 < 0$. Since the coefficient of x^2 in (21) is negative, the quadratic equation (21) has no real roots. This implies that $SOC(x)<0$ for all $x\in[0,1]$ which we want to prove.

Step 3: We shall prove that if $\alpha+\beta=4\alpha\beta$ then the maximization problem (M) has the corner solutions. To do so we first claim that, if $\alpha+\beta=4\alpha\beta$, $F(x,y)$ may achieve a unique local maximum at $(\tilde{x},\tilde{y})=(\alpha/(\alpha+\beta),\beta/(\alpha+\beta))$. However, we claim that $F(\alpha/(\alpha+\beta),\beta/(\alpha+\beta))\leq F(0,0)=F(1,1)=1$, which yields the corner solutions of (M).

We first note that, if $\alpha+\beta=4\alpha\beta$, then (22) implies that the discriminant D_1 of the quadratic equation (21) is zero for all $x\in[0,1]$. A simple calculation leads that $\tilde{x}$ is a unique solution of (21) and that $SOC(\tilde{x})=0$ and $SOC(x)<0$ for all $x\neq\tilde{x}$ as long as $(x,l(x))$ satisfies the first order condition for a local maximum of $F(x,y)$. Since $(\tilde{x},\tilde{y})$ satisfies the first order condition for a local maximum of $F(x,y)$ (Lemma 10), we conclude that $F(x,y)$ may achieve a unique local maximum at $(\tilde{x},\tilde{y})$. However, this cannot be a global maximum; i.e. $F(x,y)$ has corner solutions.

To verify that $F(x,y)$ has corner solutions when $\alpha+\beta=4\alpha\beta$, it is sufficient to show that $F(\tilde{x},\tilde{y})\leq 1$. Note from Lemma 10 that $F(\tilde{x},\tilde{y})=2\alpha^\alpha\beta^\beta/(\alpha+\beta)^{\alpha+\beta}$. If $\alpha=\beta$, then $\alpha=\beta=1/2$, i.e., $\alpha+\beta=1$; therefore, it is immediate to see that $F(\tilde{x},\tilde{y})=1=F(0,0)=F(1,1)$. We claim that, if $\alpha+\beta=4\alpha\beta$ with $\alpha\neq\beta$, then we obtain the inequality

$$F(\tilde{x},\tilde{y}) = \frac{2\alpha^\alpha\beta^\beta}{(\alpha+\beta)^{\alpha+\beta}} \leq 1. \tag{23}$$

Suppose, on the contrary, that

$$F(\tilde{x},\tilde{y}) = \frac{2\alpha^\alpha\beta^\beta}{(\alpha+\beta)^{\alpha+\beta}} > 1. \tag{24}$$

Now, set $\alpha'=\alpha+\varepsilon$, $\beta'=\beta+\varepsilon$ for $\varepsilon>0$. Then, if we take $\varepsilon>0$ small enough, we have $\alpha'+\beta'<4\alpha'\beta'$ as well as

$$\frac{2(\alpha')^{\alpha'}(\beta')^{\beta'}}{(\alpha'+\beta')^{\alpha'+\beta'}} > 1. \tag{25}$$

However, this contradicts the result of Step 2.

Step 4: We shall first prove that, if $\alpha+\beta > 4\alpha\beta$, then $F(x,y)$ achieves a unique local maximum in the interior at $(\tilde{x},\tilde{y})$; then, we shall show that the maximization problem (M) has corner solutions if and only if $2\alpha^\alpha\beta^\beta/(\alpha+\beta)^{\alpha+\beta}\leq 1$.

It should be noted from Lemma 10 that, if $\alpha+\beta>4\alpha\beta$, then $F(x,y)$ achieves a local maximum at $(\tilde{x},\tilde{y})$. Now we claim that there is no other local maximum for (M) except at $(\tilde{x},\tilde{y})$ in the interior of $[0,1]\times[0,1]$.

Suppose on the contrary, that there exists another local maximum at $(\hat{x},\hat{y})\in(0,1)\times(0,1)$ for (M). Then, it follows from Lemma 8 that $\hat{y}=l(\hat{x})$ where $l(\hat{x})\equiv\beta^2\hat{x}/(\alpha^2+(\beta^2-\alpha^2)\hat{x})$. Now, we focus our attention to those combination (x,y) satisfying the first order condition which is expressed by $(x,l(x))$ where $0\leq x\leq 1$.

Lemma 9 guarantees that the first order condition for a local maximum of $F(x,y)$ is equivalent to that for a local maximum of $L(x)=F(x,l(x))$. Therefore, we have $L'(\hat{x})=0$. For examining the graph of $L(x)$, we need the following lemmata.

Lemma 11 *Suppose that* $\alpha+\beta-4\alpha\beta>0$ *and* $F(x,y)$ *achieves a local maximum at* $(\hat{x},\hat{y})\in(0,1)\times(0,1)$ *where* $(\hat{x},\hat{y})\neq(\tilde{x},\tilde{y})$. *Given* $t\in(0,1)$, *we define* $x(t)=t\tilde{x}+(1-t)\hat{x}$ *where* $\tilde{x}=\alpha/(\alpha+\beta)$. *If* $L'(x(t))=0$ *for some* $t\in(0,1)$, *then* $F(x,y)$ *also achieves a local maximum at* $(x(t),l(x(t)))$.

Lemma 12 *If* $\alpha+\beta-4\alpha\beta>0$, *then we have*

$$\frac{d^2L(\tilde{x})}{dx^2}<0.$$

We proceed with the proof by examining two cases: $\tilde{x}<\hat{x}$; and $\tilde{x}>\hat{x}$. For the case of $\tilde{x}<\hat{x}$, since $L'(\tilde{x})=0$ and $L''(\tilde{x})<0$ (Lemma 12), there are two possibilities depending upon the sign of $L'(x)$ on an open interval $(\tilde{x},\hat{x})$.

case (a): $L'(x)>0$ for some $x\in(\tilde{x},\hat{x})$. Since $L''(\tilde{x})<0$ it follows that there is $\varepsilon'>0$ such that $L'(\tilde{x}+\varepsilon)<0$ for all ε with $0<\varepsilon<\varepsilon'$. It also follows that there exists $\varepsilon''>0$ such that $\tilde{x}+\varepsilon'<\hat{x}-\varepsilon''$, $L'(\tilde{x}+\varepsilon')<0$ and $L'(\hat{x}-\varepsilon'')>0$. In this case there is $x^*\in(\tilde{x},\hat{x})$ such that $L'(x^*)=0$ and $L''(x^*)>0$, which implies that $L(x)$ has a local minimum at x^*. Since $L'(x^*)=0$, it follows from Lemma 11 that $F(x,y)$ achieves a local maximum at $(x^*,l(x^*))$, which in turn implies that $L(x)$ is maximized at x^*, a contradiction.

case (b): $L'(x)\leq 0$ for all $x\in(\tilde{x},\hat{x})$. This implies that $L(x)$ is not increasing. If there is $x^*\in(\tilde{x},\hat{x})$ such that $L'(x^*)=0$, then it follows from Lemma 11 that $L'(x)=0$ for any $x\in(\tilde{x},\hat{x})$. However, this contradicts the fact that $L''(\tilde{x})<0$. Otherwise, we have $L'(x)<0$ for all $x \in (\tilde{x}, \hat{x})$. So, $L(x)$ is strictly decreasing at any $x \in (\tilde{x}, \hat{x})$. However, this contradicts the fact that $F(x, y)$ achieves a local maximum at $\hat{x}$.

We can prove the case of $\hat{x}<\tilde{x}$ in the same way.

So far, we have shown that, if $\alpha+\beta>4\alpha\beta$, then $F(x, y)$ achieves a unique local maximum in the interior at $(\tilde{x}, \tilde{y})$. Since $F(\tilde{x}, \tilde{y})=2\alpha^{\alpha}\beta^{\beta}/(\alpha+\beta)^{\alpha+\beta}$, a necessary and sufficient condition that $F(x, y)=x^{\alpha}y^{\beta}+(1-x)^{\beta}(1-y)^{\alpha}\leq 1$ for all $(x, y)\in[0,1]\times[0,1]$ is

$$\frac{\alpha^{\alpha}\beta^{\beta}}{(\alpha+\beta)^{\alpha+\beta}} \leq \frac{1}{2}.$$

In other words, the maximization problem (M) has corner solutions if and only if $2\alpha^{\alpha}\beta^{\beta}/(\alpha+\beta)^{\alpha+\beta}\leq 1$. **Q.E.D.**

To prove Corollary 6 and Corollary 7, we need the following two lemmata, which will be proved in Section 4.

Lemma 13 *Let* $A_1, A_2>0$ *and* $\bar{x}, \bar{y}>0$. *If, for all* $(x, y)\in[0,1]\times[0,1]$,

$$x^{\alpha_1}y^{\beta_1}+(1-x)^{\alpha_2}(1-y)^{\beta_2} \leq 1 \tag{26}$$

then, we have, for all $(x, y)\in[0, \bar{x}]\times[0, \bar{y}]$,

$$A_1x^{\alpha_1}y^{\beta_1}+A_2(\bar{x}-x)^{\alpha_2}(\bar{y}-y)^{\beta_2} \leq \max\{A_1\bar{x}^{\alpha_1}\bar{y}^{\beta_1}, A_2\bar{x}^{\alpha_2}\bar{y}^{\beta_2}\}. \tag{27}$$

If we establish the condition (26) for a very specific case that $A_1=A_2=1$ and $\bar{x}=\bar{y}=1$, this lemma guarantees the conclusion (27) for general cases of $A_1, A_2>0$ and $\bar{x}, \bar{y}>0$.

In addition to Lemma 13, we need the following lemma to prove Corollary 7.

Lemma 14 *Let* $\alpha, \beta>0$ *with* $\alpha+\beta\geq 1$. *If, for all* $(x, y)\in[0,1]\times[0,1]$, *we have*

$$x^{\alpha}y^{\beta}+(1-x)^{\beta}(1-y)^{\alpha} \leq 1 \tag{28}$$

then, for any $\varepsilon_i\geq 0\,(i=1,2,3,4)$,

$$x^{\alpha+\varepsilon_1}y^{\beta+\varepsilon_2}+(1-x)^{\beta+\varepsilon_3}(1-y)^{\alpha+\varepsilon_4} \leq 1 \tag{29}$$

for all $(x, y)\in[0,1]\times[0,1]$.

If we establish the condition (28) for a symmetric case with $A_1=A_2=1$ and $\bar{x}=\bar{y}=1$, this lemma guarantees the conclusion (29) for coefficients greater than α and β.

Let us now proceed to the proofs of Corollary 6 and Corollary 7.

Proof of Corollary 6

Suppose that $2\alpha^{\alpha}\beta^{\beta}/(\alpha+\beta)^{\alpha+\beta}\leq 1$ is satisfied. Then, it follows from Theorem 5 that we have, for all $(x, y)\in[0, 1]\times[0, 1]$,

$$x^{\alpha}y^{\beta}+(1-x)^{\beta}(1-y)^{\alpha}\leq 1.$$

This is equivalent to (26) in Lemma 13 for the case of $\alpha_1=\beta_2=\alpha$ and $\alpha_2=\beta_1=\beta$. Thus, Lemma 13 yields

$$A_1x^{\alpha}y^{\beta}+A_2(\bar{x}-x)^{\beta}(\bar{y}-y)^{\alpha}\leq \max(A_1\bar{x}^{\alpha}\bar{y}^{\beta}, A_2\bar{x}^{\beta}\bar{y}^{\alpha}).$$

Q.E.D.

Proof of Corollary 7

Suppose that $2\alpha^{\alpha}\beta^{\beta}/(\alpha+\beta)^{\alpha+\beta}\leq 1$ is satisfied. The, it follows from Theorem 5 and Lemma 14 that we have, for any $\varepsilon_i>0\,(i=1, 2, 3, 4)$,

$$x^{\alpha+\varepsilon_1}y^{\beta+\varepsilon_2}+(1-x)^{\beta+\varepsilon_3}(1-y)^{\alpha+\varepsilon_4}\leq 1$$

for all $(x, y)\in[0, 1]\times[0, 1]$. Now, set $\alpha_1=\alpha+\varepsilon_1$, $\beta_1=\beta+\varepsilon_2$, $\alpha_2=\beta+\varepsilon_3$ and $\beta_2=\alpha+\varepsilon_4$. Then, we have

$$x^{\alpha_1}y^{\beta_1}+(1-x)^{\alpha_2}(1-y)^{\beta_2}\leq 1.$$

Therefore, Lemma 13 leads to the following inequality:

$$A_1x^{\alpha_1}y^{\beta_1}+A_2(\bar{x}-x)^{\alpha_2}(\bar{y}-y)^{\beta_2}\leq \max\{A_1\bar{x}^{\alpha_1}\bar{y}^{\beta_1}, A_2\bar{x}^{\alpha_2}\bar{y}^{\beta_2}\}.$$

Q.E.D.

4 Proofs of Lemmata

Proof of Lemma 8

See the proof of Lemma 2 in Hamano (2011a). **Q.E.D.**

Proof of Lemma 9

Note first that, since $L(x)=F(x,l(x))=x^{\alpha}l(x)^{\beta}+(1-x)^{\beta}(1-l(x))^{\alpha}$, we have

$$\frac{dL(x)}{dx} = F_x(x,l(x))+F_y(x,l(x))\times l'(x). \tag{30}$$

This immediately shows that $F_x(x,l(x))=F_y(x,l(x))=0$ implies $dL(x)/dx=0$.

For proving that $dL(x)/dx=0$ implies $F_x(x,l(x))=F_y(x,l(x))=0$, we first claim that

$$F_y(x,l(x)) = \frac{\alpha^2+(\beta^2-\alpha^2)x}{\alpha\beta}\times F_x(x,l(x)). \tag{31}$$

To show this, from the first order condition and $l(x)=\beta^2x/(\alpha^2+(\beta^2-\alpha^2)x)$, we obtain

$$\begin{aligned}
F_x(x,l(x)) &= \alpha x^{\alpha-1}(l(x))^{\beta}-\beta(1-x)^{\beta-1}(1-l(x))^{\alpha}\\
&= \alpha x^{\alpha-1}\left(\frac{\beta^2x}{\alpha^2+(\beta^2-\alpha^2)x}\right)^{\beta}-\beta(1-x)^{\beta-1}\left(\frac{\alpha^2(1-x)}{\alpha^2+(\beta^2-\alpha^2)x}\right)^{\alpha}\\
&= \frac{\alpha\beta^{2\beta}x^{\alpha+\beta-1}}{\{\alpha^2+(\beta^2-\alpha^2)x\}^{\beta}}-\frac{\alpha^{2\alpha}\beta(1-x)^{\alpha+\beta-1}}{\{\alpha^2+(\beta^2-\alpha^2)x\}^{\alpha}}.
\end{aligned}$$

Using this, we have

$$\begin{aligned}
F_y(x,l(x)) &= \beta x^{\alpha}l(x)^{\beta-1}-\alpha(1-x)^{\beta}(1-l(x))^{\alpha-1}\\
&= \beta x^{\alpha}\left(\frac{\beta^2x}{\alpha^2+(\beta^2-\alpha^2)x}\right)^{\beta-1}-\alpha(1-x)^{\beta}\left(1-\frac{\beta^2x}{\alpha^2+(\beta^2-\alpha^2)x}\right)^{\alpha-1}\\
&= \frac{\beta^{2\beta-1}x^{\alpha+\beta-1}}{\{\alpha^2+(\beta^2-\alpha^2)x\}^{\beta-1}}-\frac{\alpha^{2\alpha-1}(1-x)^{\alpha+\beta-1}}{\{\alpha^2+(\beta^2-\alpha^2)x\}^{\alpha-1}}\\
&= \frac{\alpha^2+(\beta^2-\alpha^2)x}{\alpha\beta}\left\{\frac{\alpha\beta^{2\beta}x^{\alpha+\beta-1}}{\{\alpha^2+(\beta^2-\alpha^2)x\}^{\beta}}-\frac{\alpha^{2\alpha\beta(1-x)^{\alpha+\beta-1}}}{\{\alpha^2+(\beta^2-\alpha^2)x\}^{\alpha}}\right\}\\
&= \frac{\alpha^2+(\beta^2-\alpha^2)x}{\alpha\beta}F_x(x,l(x)).
\end{aligned}$$

This shows that the claim is true. We also note that

$$l'(x) = \frac{\alpha^2\beta^2}{\{\alpha^2+(\beta^2-\alpha^2)x\}^2}. \tag{32}$$

Now, it follows from (30), (31) and (32) that

$$\begin{aligned}
\frac{dL(x)}{dx} &= F_x(x,l(x))\left\{1+\frac{\alpha^2+(\beta^2-\alpha^2)x}{\alpha\beta}\times\frac{\alpha^2\beta^2}{\{\alpha^2+(\beta^2-\alpha^2)x\}^2}\right\}\\
&= F_x(x,l(x))\times(\alpha+\beta)\times\frac{(1-x)\alpha+\beta x}{(1-x)\alpha^2+\beta^2}.
\end{aligned} \tag{33}$$

Thus, we conclude that $dL(x)/dx=0$ implies $F_x(x, l(x))=F_y(x, l(x))=0$. **Q.E.D.**

Proof of Lemma 10

First of all, it is immediate to see that $F_x(\tilde{x}, \tilde{y})=F_y(\tilde{x}, \tilde{y})=0$. It is also easy to derive the relation (18).

To check the relationship (19), we have,

$$F_{xx}(\tilde{x}, \tilde{y}) = \alpha(\alpha-1)\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha-2}\left(\frac{\beta}{\alpha+\beta}\right)^{\beta}+\beta(\beta-1)\left(\frac{\beta}{\alpha+\beta}\right)^{\beta-2}\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha}$$

$$= \frac{\alpha^{\alpha-1}\beta^{\beta-1}}{(\alpha+\beta)^{\alpha+\beta-2}}(2\alpha\beta-\alpha-\beta),$$

and

$$F_{yy}(\tilde{x}, \tilde{y}) = \beta(\beta-1)\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha}\left(\frac{\beta}{\alpha+\beta}\right)^{\beta-2}+\alpha(\alpha-1)\left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha-2}$$

$$= F_{xx}(\tilde{x}, \tilde{y}).$$

To see the expression (20), we obtain

$$SOC(\tilde{x}) = -(\alpha-\beta)(\alpha^2-\beta^2)\left(\frac{\alpha}{\alpha+\beta}\right)^2+(\alpha^2-\beta^2)(2\alpha-1)\left(\frac{\alpha}{\alpha+\beta}\right)+(1-\alpha-\beta)\alpha^2$$

$$= \frac{\alpha\beta}{\alpha+\beta}(\alpha+\beta-4\alpha\beta).$$

Finally, if $\alpha+\beta>4\alpha\beta$, note that

$$F_{xx}(\tilde{x}, \tilde{y}) = F_{yy}(\tilde{x}, \tilde{y}) = \frac{\alpha^{\alpha-1}\beta^{\beta-1}}{(\alpha+\beta)^{\alpha+\beta-2}}(2\alpha\beta-\alpha-\beta) < \frac{\alpha^{\alpha-1}\beta^{\beta-1}}{(\alpha+\beta)^{\alpha+\beta-2}}(4\alpha\beta-\alpha-\beta) < 0,$$

and

$$SOC(\tilde{x}) = \frac{\alpha\beta}{\alpha+\beta}(\alpha+\beta-4\alpha\beta) > 0.$$

Since $F(x, y)$ satisfies the first order condition for a local maximum at $(\tilde{x}, \tilde{y})$, it follows from the relationship (16) in Lemma 8 that

$$F_{xx}(\tilde{x}, \tilde{y})F_{yy}(\tilde{x}, \tilde{y})-(F_{xy}(\tilde{x}, \tilde{y}))^2 < 0.$$

Therefore, we conclude that, if $\alpha+\beta>4\alpha\beta$, $F(x, y)$ achieves a local maximum at the point $(\tilde{x}, \tilde{y})$. **Q.E.D.**

Proof of Lemma 11

Suppose that $L'(x(t^*))=0$ for some $t^* \in (0, 1)$. Let us denote $x^*=x(t^*)$ and $y^*=l(x(t^*))$. Then, it suffices to show that

$$F_x(x^*, y^*) = 0, \quad F_x(x^*, y^*) = 0, \tag{34}$$

$$F_{xx}(x^*, y^*) < 0, \quad F_{yy}(x^*, y^*) < 0, \tag{35}$$

and

$$F_{xx}(x^*, y^*)F_{yy}(x^*, y^*) - \{F_{yy}(x^*, y^*)\}^2 > 0. \tag{36}$$

First, it follows from Lemma 9 that $L'(x^*)=0$ implies the condition (34).

To verify the condition (35), note that if (x, y) satisfies the first order condition, then we have[5]

$$F_{xx}(x, y) = \frac{\alpha x^{\alpha-2} y^{\beta}}{1-x}\{(\alpha-1)+(\beta-\alpha)x\},$$

$$F_{yy}(x, y) = \frac{\beta x^{\alpha} y^{\beta-2}}{1-y}\{(\beta-1)+(\alpha-\beta)y\}.$$

Now, since $F(x, y)$ attains a local maximum at $(\hat{x}, \hat{y})$, then we have $F_{xx}(\hat{x}, \hat{y}) \leq 0$, which implies $(\alpha-1)+(\beta-\alpha)\hat{x} \leq 0$. Note also that $F_{xx}(\tilde{x}, \tilde{y})<0$, which implies $(\alpha-1)+(\beta-\alpha)\tilde{x}<0$. Therefore, since $x^*=t^*\tilde{x}+(1-t^*)\hat{x}$ with $t^* \in (0, 1)$, we conclude that $(\alpha-1)+(\beta-\alpha)x^*<0$, which yields $F_{xx}(x^*, y^*)<0$. Similarly, we have $F_{yy}(x^*, y^*)<0$.

Finally, for (36) it suffices to show that $SOC(x^*)>0$. Since the function $SOC(x)$ is quadratic and convex, the range of x satisfying $SOC(x)>0$ is an interval. Moreover, Lemma 8 yields $SOC(\tilde{x})>0$ and $SOC(\hat{x}) \geq 0$. Therefore, $SOC(x^*)>0$ follows from the fact that x^* is a convex combination of $\tilde{x}$ and $\hat{x}$. **Q.E.D.**

Proof of Lemma 12 Since $F_y(\tilde{x}, l(\tilde{x}))=F_y(\tilde{x}, \tilde{y})=0$, we have

$$\frac{d^2L(\tilde{x})}{dx^2} = F_{yy}(\tilde{x}, l(\tilde{x}))\{l'(\tilde{x})\}^2+2F_{xy}(\tilde{x}, l(\tilde{x}))l'(\tilde{x})+F_{xx}(\tilde{x}, l(\tilde{x})).$$

If we set the right-hand side of this to zero, we obtain a quadratic equation of $l'(x)$, the discriminant D_2 of which is calculated as follows:

$$\begin{aligned} D_2 &= 4(F_{xy}(\tilde{x}, l(\tilde{x})))^2-4F_{xx}(\tilde{x}, l(\tilde{x}))F_{yy}(\tilde{x}, l(\tilde{x})) \\ &= -4\{F_{xx}(\tilde{x}, l(\tilde{x}))F_{yy}(\tilde{x}, l(\tilde{x}))-(F_{xy}(\tilde{x}, l(\tilde{x})))^2\}. \end{aligned}$$

It follows from the assumption $\alpha+\beta-4\alpha\beta>0$ that $F_{xx}(\tilde{x}, l(\tilde{x}))F_{yy}(\tilde{x}, l(\tilde{x}))-(F_{xy}(\tilde{x}, l(\tilde{x})))^2>0$; therefore, $D_2<0$. Therefore, noting that $F_{yy}(\tilde{x}, l(\tilde{x}))<0$, we have $d^2L/dx^2<0$ for all $l'(\tilde{x})$, which leads to the desired inequality. **Q.E.D.**

Proof of Lemma 13

Suppose, on the contrary, that there exists a vector $(\hat{x}, \hat{y})\in[0, \bar{x}]\times[0, \bar{y}]$ such that

$$\max\{A_1\bar{x}^{\alpha_1}\bar{y}^{\beta_1}, A_2\bar{x}^{\alpha_2}\bar{y}^{\beta_2}\} < A_1\hat{x}^{\alpha_1}\hat{y}^{\beta_1}+A_2(\bar{x}-\hat{x})^{\alpha_2}(\bar{y}-\hat{y})^{\beta_2}. \tag{37}$$

We divide the proof into two cases: (i) $A_1\bar{x}^{\alpha_1}\bar{y}^{\beta_1}\geq A_2\bar{x}^{\alpha_2}\bar{y}^{\beta_2}$ and (ii) $A_1\bar{x}^{\alpha_1}\bar{y}^{\beta_1}<A_2\bar{x}^{\alpha_2}\bar{y}^{\beta_2}$.

We shall now prove the case (i) $A_1\bar{x}^{\alpha_1}\bar{y}^{\beta_1}\geq A_2\bar{x}^{\alpha_2}\bar{y}^{\beta_2}$. Then, this yields the following relationship:

$$\frac{A_2}{A_1\bar{x}^{\alpha_1}\bar{y}^{\beta_1}} \leq \frac{1}{\bar{x}^{\alpha_2}\bar{y}^{\beta_2}}. \tag{38}$$

Now, it follows from (37) that

$$A_1\bar{x}^{\alpha_1}\bar{y}^{\beta_1} < A_1\hat{x}^{\alpha_1}\hat{y}^{\beta_1}+A_2(\bar{x}-\hat{x})^{\alpha_2}(\bar{y}-\hat{y})^{\beta_2}.$$

Dividing both sides of this inequality by $A_1\bar{x}^{\alpha_1}\bar{y}^{\beta_1}$, we have

$$\begin{aligned} 1 &< \left(\frac{\hat{x}}{\bar{x}}\right)^{\alpha_1}\left(\frac{\hat{y}}{\bar{y}}\right)^{\beta_1}+\frac{A_2}{A_1\bar{x}^{\alpha_1}\bar{y}^{\beta_1}}(\bar{x}-\hat{x})^{\alpha_2}(\bar{y}-\hat{y})^{\beta_2} \\ &\leq \left(\frac{\hat{x}}{\bar{x}}\right)^{\alpha_1}\left(\frac{\hat{y}}{\bar{y}}\right)^{\beta_1}+\frac{1}{\bar{x}^{\alpha_2}\bar{y}^{\beta_2}}(\bar{x}-\hat{x})^{\alpha_2}(\bar{y}-\hat{y})^{\beta_2} \\ &= \left(\frac{\hat{x}}{\bar{x}}\right)^{\alpha_1}\left(\frac{\hat{y}}{\bar{y}}\right)^{\beta_1}+\left(\frac{\bar{x}-\hat{x}}{\bar{x}}\right)^{\alpha_2}\left(\frac{\bar{y}-\hat{y}}{\bar{y}}\right)^{\beta_2} \\ &= \left(\frac{\hat{x}}{\bar{x}}\right)^{\alpha_1}\left(\frac{\hat{y}}{\bar{y}}\right)^{\beta_1}+\left(1-\frac{\hat{x}}{\bar{x}}\right)^{\alpha_2}\left(1-\frac{\hat{y}}{\bar{y}}\right)^{\beta_2}. \end{aligned}$$

Note that the second inequality follows from (38). However, this contradicts the hypothesis since $(\hat{x}/\bar{x}, \hat{y}/\bar{y})\in[0, 1]\times[0, 1]$.

It is clear that we can prove the case (ii) in the same way. **Q.E.D.**

Proof of Lemma 14

Suppose not, *i.e.*, there exists $(\hat{x}, \hat{y})\in[0, \bar{x}]\times[0, \bar{y}]$ and $\hat{\varepsilon}_i\geq 0(i=1, 2, 3, 4)$ such that

$$1 < \hat{x}^{\alpha+\hat{\varepsilon}_1}\hat{y}^{\beta+\hat{\varepsilon}_2}+(1-\hat{x})^{\beta+\hat{\varepsilon}_3}(1-\hat{y})^{\alpha+\hat{\varepsilon}_4}. \tag{39}$$

Denoting by A the right-hand side of this inequality we have

$$\begin{aligned} A &= \hat{x}^{\hat{\varepsilon}_1}\hat{y}^{\hat{\varepsilon}_2}\times\hat{x}^{\alpha}\hat{y}^{\beta}+(1-\hat{x})^{\hat{\varepsilon}_3}(1-\hat{y})^{\hat{\varepsilon}_4}(1-\hat{x})^{\beta}(1-\hat{y})^{\alpha} \\ &\leq \max\{\hat{x}^{\hat{\varepsilon}_1}\hat{y}^{\hat{\varepsilon}_2}, (1-\hat{x})^{\hat{\varepsilon}_3}(1-\hat{y})^{\hat{\varepsilon}_4}\}\times\{\hat{x}^{\alpha}\hat{y}^{\beta}+(1-\hat{x})^{\beta}(1-\hat{y})^{\alpha}\}. \end{aligned} \tag{40}$$

Now, we claim that $\max\{\hat{x}^{\hat{\varepsilon}_1}\hat{y}^{\hat{\varepsilon}_2}, (1-\hat{x})^{\hat{\varepsilon}_3}(1-\hat{y})^{\hat{\varepsilon}_4}\}\leq 1$. First note that, since $0\leq\hat{x}, \hat{y}\leq 1$, $0<1-\hat{x}\leq 1$ and $0<1-\hat{y}\leq 1$, we have $\hat{x}^{\hat{\varepsilon}_1}$, $\hat{y}^{\hat{\varepsilon}_2}$, $(1-\hat{x})^{\hat{\varepsilon}_3}$, $(1-\hat{y})^{\hat{\varepsilon}_4}\leq 1$, which yields $\hat{x}^{\hat{\varepsilon}_1}\hat{y}^{\hat{\varepsilon}_2}\leq 1$ and $(1-\hat{x})^{\hat{\varepsilon}_3}(1-\hat{y})^{\hat{\varepsilon}_4}\leq 1$, from which the claim follows. Therefore, we have

$$1 < A \leq \hat{x}^{\alpha}\hat{y}^{\beta}+(1-\hat{x})^{\beta}(1-\hat{y})^{\alpha} \leq 1,$$

a contradiction. **Q.E.D.**

5 Example and Illustration

In this section we provide an illustrative example concerning the one in which the condition in Theorem 5 is satisfied but the condition in Hamano (2011a) is violated.

Example 15 *Let us consider the following two symmetric generalized Cobb-Douglas production functions:*

$$\begin{aligned} f^1(x, y) &= x^{0.25}y^3, \\ f^2(x, y) &= x^3y^{0.25}. \end{aligned}$$

This is the case of $\alpha=0.25$ and $\beta=3$ in Theorem 5. Then, we have

$$\frac{\alpha^{\alpha}\beta^{\beta}}{(\alpha+\beta)^{\alpha+\beta}} \approx 0.4142 < \frac{1}{2}.$$

According to Theorem 5 we conclude that for any $(\bar{x}, \bar{y})\gg(0, 0)$ monopoly production is efficient. Note that $\alpha+\beta=3.25>3=4\alpha\beta$. Therefore, the condition in Hamano (2011a) is not satisfied.

In Figure 1 the graphs of $\alpha+\beta=4\alpha\beta$ and $\alpha^{\alpha}\beta^{\beta}/(\alpha+\beta)^{\alpha+\beta}=1/2$ are drawn. Note that the graph of $\alpha+\beta=4\alpha\beta$ is located in the upper-right side of the graph of $\alpha^{\alpha}\beta^{\beta}/(\alpha+\beta)^{\alpha+\beta}=1/2$ except at $(1/2, 1/2)$ in the (α, β)-plane. As is depicted, the combination of $(\alpha, \beta)=(0.25, 3)$ is on the region between these two graphs.

In Figure 2 we draw the graphs of $z=x^{\alpha}l(x)^{\beta}+(1-x)^{\beta}(1-l(x))^{\alpha}$ for various values of β when $\alpha=3$. Note that the function $l(x)$ is derived from the first order condition in Step 1 of the proof of Theorem 5 so that a local maximum for $x^{\alpha}y^{\beta}+(1-x)^{\beta}(1-y)^{\alpha}$ must be achieved

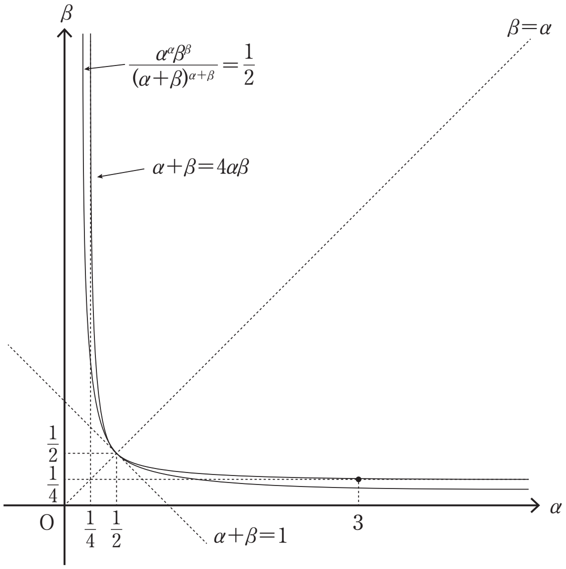

Figure 1: Graphs of $\alpha^{\alpha}\beta^{\beta}/(\alpha+\beta)^{\alpha+\beta}=1/2$ and $\alpha+\beta=4\alpha\beta$

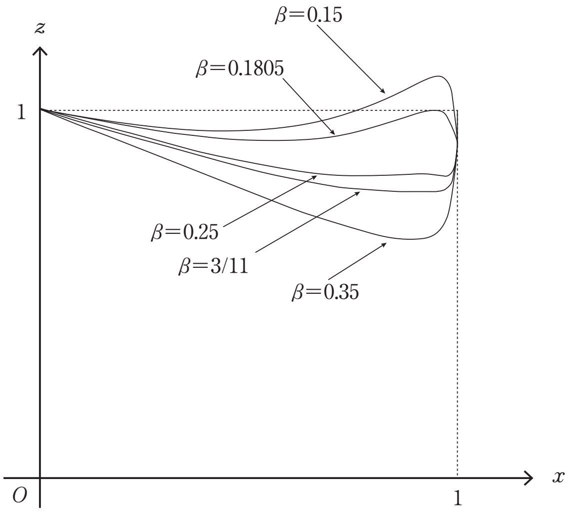

Figure 2: Graphs of $z=x^{\alpha}l(x)^{\beta}+(1-x)^{\beta}(1-l(x))^{\alpha}$
for various values of β when $\alpha=3$ where $l(x)=\beta^2x/(\alpha^2+(\beta^2-\alpha^2)x)$

at some $(x, l(x))$. Notice also that when $\alpha=3$ and $\beta=0.1805$, we have

$$\frac{\alpha^{\alpha}\beta^{\beta}}{(\alpha+\beta)^{\alpha+\beta}} \approx \frac{1}{2}.$$

Therefore, the graph of $z=x^{\alpha}l(x)^{\beta}+(1-x)^{\beta}(1-l(x))^{\alpha}$ for $\alpha=3$ and $\beta=0.1805$ is approximately tangent at $\alpha/(\alpha+\beta)\approx 0.94325$ to the horizontal line $z=1$. That is, if $\tilde{x}=0.94325$, then $\tilde{x}^{\alpha}l(\tilde{x})^{\beta}+(1-\tilde{x})^{\beta}(1-l(\tilde{x}))^{\alpha}\approx 1=\max[f^1(1,1),f^2(1,1)]$ for $\alpha=3$ and $\beta=0.1805$, which implies that sum of outputs of two technologies is equal to that produced by monopoly.

Note also that when $\alpha=3$ and $\beta=3/11$, we have $\alpha+\beta=4\alpha\beta$. Thus, the function $z=x^{\alpha}l(x)^{\beta}+(1-x)^{\beta}(1-l(x))^{\alpha}$ for $\alpha=3$ and $\beta=3/11$ has no local maximum between 0 and 1 according to Step 2 of the proof of Theorem 5.

6 Concluding Remark

In this paper we consider an economy with two increasing returns technologies expressed by generalized Cobb-Douglas production functions and examine the problem on aggregation of these technologies. We first provide, for two symmetric generalized Cobb-Douglas production functions with equal amounts of available inputs, a necessary and sufficient condition for monopoly to lead aggregate production efficiency, i.e., corner solutions for non-convex maximization problem. Then, as corollaries, we derive sufficient conditions for efficiency of monopoly production for a wider class of two generalized Cobb-Douglas production functions.

Our result is certainly new in the sense that we derive a solution to non-convex optimization problem. However, Theorem 5 crucially depends upon the symmetric property of two non-convex functions. If we try to generalize the model to the case with three or more inputs, this property breaks down. Therefore, it is not clear how to extend our results to more general settings such as the case of arbitrary number of inputs or arbitrary coefficients of generalized Cobb-Douglas production functions.

Acknowledgement

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Notes

1) See Mas-Colell et al. (1995, pp. 147-149) for example.
2) See Beato and Mas-Colell (1985) for example.
3) There is not much research on these topics. A few exception is Ginsberg (1974) that derived a sufficient condition that guarantees aggregate production efficiency in a case of one input and one output where production functions are expressed by nicely convex-concave ones. Recently,

Ceparano and Quartieri (2019) clarified the condition leading to convexity of the aggregate production set even if individual production sets may not be convex. In a broader sense their result can be regarded as deriving the condition on aggregate production efficiency.

4) As will be clear in Section 5, $F(x, y)$ achieves a unique local minimum at $(\tilde{x}, \tilde{y})$

5) These are equivalent to (14) and (15) in Hamano (2011a).

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