Critical Discount Factors for Maximal Collusion in Cournot Oligopoly Supergames

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Abstract

We consider n-firm symmetric Cournot oligopoly supergames where the inverse demand function is linear and the firms' cost functions are quadratic with non-decreasing marginal costs. For each n, we compute the smallest common discount factor at which full collusion among the firms is sustained by some subgame perfect equilibrium with symmetric punishments. This discount factor is also the smallest discount factor at which the full collusion is sustained by some subgame perfect equilibrium, regardless of whether the punishments are symmetric or asymmetric, if and only if n is greater than some specific number.

1. Introduction

In oligopoly theory, collusive behaviour is often explained by using infinitely repeated games with discounting (supergames) where a deviant firm is "punished" according to some punishment mechanism. In his classic work, Friedman (1971) introduced trigger strategies which entail a punishment by moving to the Nash equilibrium of the constituent game indefinitely. On the other hand, in the context of symmetric Cournot oligopoly supergames, Abreu (1986) introduced stick-and-carrot strategies. In these strategies, if a firm deviates, then all firms including the deviant firm choose large outputs for just one period to punish each other, and then revert to the original collusion.

In the present paper we consider n-firm symmetric Cournot oligopoly supergames, where the inverse demand function is linear and the firms' (identical) cost functions are quadratic with non-decreasing (possibly constant) marginal costs. For each n, we compute the smallest common discount factor at which full collusion among the firms is sustained by some subgame perfect stick-and-carrot strategy profile. Abreu's results imply that such a discount factor is the smallest discount factor at which the full collusion is sustained by some subgame perfect equilibrium with symmetric punishments.[1)] It further follows from the results in Abreu (1986) and Segerstrom (1991) that it is the smallest discount factor at which the full collusion is sustained by some subgame perfect equilibrium, regardless of whether the punishments are symmetric or asymmetric, if and only if n is greater than some specific number. Finding the smallest common discount factor is of interest, especially from the viewpoint of the stability of collusion.

The paper proceeds as follows. Section 2 presents the model, and Section 3 presents the results.

2. The Model

We consider an industry producing a homogeneous commodity with a given number of firms $n \geq 2$. All firms have identical cost functions and face an inverse demand function. We denote the cost function of firm i by $C(q_i)$, and the inverse demand function by $F(Q)$, where q_i is the output of firm i and $Q = \sum_{k=1}^{n} q_k$. We assume that

$$C(q_i) = cq_i + eq_i^2 \quad \text{for all } q_i \in [0, \infty), \qquad F(Q) = a - bQ \quad \text{for all } Q \in [0, \infty),$$

with $a > c > 0$, $b > 0$, $e \geq 0$. The profit of firm i is, therefore,

$$\pi(q_i, X) \equiv F(q_i + X)q_i - C(q_i) = (a - c - bX)q_i - (b + e)q_i^2,$$

where $X = \sum_{k \neq i} q_k$.

The above situation defines a strategic form game with n players (firms), in which the strategy of player i is $q_i \in [0, \infty)$ and her payoff is $\pi(q_i, \sum_{k \neq i} q_k)$. We denote this n-player game by G.

Let $q^*(X) \equiv \arg\max_{q_i} \pi(q_i, X)$. $q^*(X)$ is the reaction function. It is simple to check that

$$q^*(X) = \max\{(a - c - bX)/2(b + e), \quad 0\}, \quad \pi^*(X) \equiv \max_{q_i} \pi(q_i, X) = (b + e)q^*(X)^2. \tag{1}$$

A Cournot-Nash equilibrium in G is an output vector $(\hat{q}_1, \cdots, \hat{q}_n)$ such that, for each firm i, $q_i = \hat{q}_i$ maximizes $\pi(q_i, \sum_{k \neq i} \hat{q}_k)$. It follows from (1) that, for all i, $\hat{q}_i = (a - c)/[b(n + 1) + 2e]$ and $\pi(\hat{q}_i, \sum_{k \neq i} \hat{q}_k) = (a - c)^2(b + e)/[b(n + 1) + 2e]^2$. We denote the Cournot-Nash equilibrium output and the corresponding profit of each firm by q^c and π^c, respectively. Thus,

$$q^c = \frac{(a-c)}{b(n+1)+2e}, \qquad \pi^c = \frac{(a-c)^2(b+e)}{[b(n+1)+2e]^2}. \tag{2}$$

If all firms collude to maximize total industry profit, the output of each firm and the corresponding profit are

$$q^m \equiv \frac{(a-c)}{2(bn+e)}, \qquad \pi^m \equiv \frac{(a-c)^2}{4(bn+e)}. \tag{3}$$

Here, if $e=0$ (i.e., if marginal cost is constant), we assume that the total industry output is divided equally among the firms.

If a firm deviates from the above collusion, it follows from (1) that its profit-maximizing output and profit are

$$q^{m*} \equiv q^*((n-1)q^m) = \frac{(a-c)[b(n+1)+2e]}{4(bn+e)(b+e)},$$

$$\pi^{m*} \equiv \pi^*((n-1)q^m) = (b+e)(q^{m*})^2 = \frac{(a-c)^2[b(n+1)+2e]^2}{16(bn+e)^2(b+e)}. \tag{4}$$

We next describe the repeated game in which the game G is repeated infinitely often and all firms face the common discount factor $\delta \in (0,1)$. We denote this game by $G^\infty(\delta)$.[2)]

Let $q(t)$ be an output vector at period t. A stream of output vectors $Q \equiv \{q(t)\}_{t=1}^\infty$ is called an *outcome path* or *punishment*. A strategy profile in $G^\infty(\delta)$ generates an outcome path inductively. Let $(Q^0, Q^1, \cdots, Q^n)$ be an $(n+1)$-vector of outcome paths associated with the following strategy profile: The firms initially play outcome path Q^0 until some firm deviates from Q^0; if firm i deviates from Q^0 then the firms play punishment Q^i until some firm deviates; if firm j deviates from Q^i then the firms play punishment Q^j until some firm deviates; and so on. Such a strategy profile is called a *simple strategy profile*. Abreu (1988, Proposition 5) proved that Q^0 is the outcome of a subgame perfect equilibrium if and only if it is the outcome of some subgame perfect simple strategy profile. Thus, to analyze the outcomes of subgame perfect equilibria, we can restrict attention to simple strategy profiles. In what follows, we will identify a simple strategy profile with the associated vector of outcome paths $(Q^0, Q^1, \cdots, Q^n)$.

Let $Q^k \equiv \{q^k(t)\}_{t=1}^\infty$ for $k=0, 1, \cdots, n$. If $q^0(t)=(q^m, q^m, \cdots, q^m)$ for all t, the total industry profit is maximized at all periods of the initial outcome path Q^0. In this case, we call Q^0 the fully collusive outcome path, or simply, the *maximal collusion*.

In the present context where our focus is on the maximal collusion, a *trigger strategy profile* can be defined by a vector of outcome paths $(Q^0, Q^1, \cdots, Q^n)$ such that $q^0(t) = (q^m, q^m, \cdots, q^m)$ for all t, and for each $k=1, \cdots, n$, $q^k(t)=(q^c, q^c, \cdots, q^c)$ for all t. It can be verified that the trigger strategy profile is a subgame perfect equilibrium in $G^{\infty}(\delta)$ if and only if

$$\delta \geq \delta^{T} \equiv \frac{\pi^{m*}-\pi^{m}}{\pi^{m*}-\pi^{c}}.^{3)} \tag{5}$$

This means that the maximal collusion is the outcome of the subgame perfect trigger strategy profile in $G^{\infty}(\delta)$ if and only if $\delta \geq \delta^{T}$.

In the context of symmetric Cournot oligopoly supergames, Abreu (1986) introduced stick-and-carrot strategies. In the present context, a (symmetric) ***stick-and-carrot strategy profile*** with stick $\bar{q}$ is defined by a vector of outcome paths $(Q^{0}, Q^{1}, \cdots, Q^{n})$ such that $q^{0}(t)=(q^{m}, q^{m}, \cdots, q^{m})$ for all t, and for each $k=1, \cdots, n$, $q^{k}(1)=(\bar{q}, \bar{q}, \cdots, \bar{q})$ and $q^{k}(t)=(q^{m}, q^{m}, \cdots, q^{m})$ for all $t \geq 2$.

Let

$$\bar{\pi} \equiv \pi(\bar{q}, (n-1)\bar{q}), \qquad \bar{\pi}^{*} \equiv \pi^{*}((n-1)\bar{q}).$$

Then the following conditions are necessary and sufficient for a stick-and-carrot strategy profile with stick $\bar{q}$ to be a subgame perfect equilibrium in $G^{\infty}(\delta)$:

$$\pi^{m}+\delta\pi^{m} \geq \pi^{m*}+\delta\bar{\pi}, \qquad \bar{\pi}+\delta\pi^{m} \geq \bar{\pi}^{*}+\delta\bar{\pi}. \tag{6}$$

The former inequality means that one-shot deviations from the collusive phase are not profitable. The latter inequality means that one-shot deviations from the punishment phase are not profitable.[4)]

We are interested in the smallest discount factor satisfying (6). We call such a discount factor the *critical discount factor*. To compute it, define the functions $f(\bar{q})$ and $g(\bar{q})$ for $\bar{q} \in [q^{c}, \infty)$ by

$$f(\bar{q}) \equiv (\pi^{m*}-\pi^{m})/(\pi^{m}-\bar{\pi}), \qquad g(\bar{q}) \equiv (\bar{\pi}^{*}-\bar{\pi})/(\pi^{m}-\bar{\pi}).$$

$f(\bar{q})$ and $g(\bar{q})$ are the values of δ satisfying $\pi^{m}+\delta\pi^{m}=\pi^{m*}+\delta\bar{\pi}$ and $\bar{\pi}+\delta\pi^{m}=\bar{\pi}^{*}+\delta\bar{\pi}$, respectively. Then $f(\bar{q})$ and $g(\bar{q})$ are continuous on $[q^{c}, \infty)$, $f(q^{c})>0=g(q^{c})$, $f'(\bar{q})<0$ for all $\bar{q}>q^{c}$, $g(\bar{q}) \in (0, 1)$ for all $\bar{q}>q^{c}$, $f(\bar{q}) \to 0$ as $\bar{q} \to \infty$, and $g(\bar{q}) \to 1$ as $\bar{q} \to \infty$. Moreover, it can be shown that $g(\bar{q})$ is strictly increasing for all $\bar{q}>q^{c}$.[5)] Let

$$\delta^{S} \equiv \min\{\delta \in (0, 1) \mid \delta \geq f(\bar{q}) \text{ and } \delta \geq g(\bar{q}) \text{ for some } \bar{q} \in [q^{c}, \infty)\}.$$

Clearly, δ^{S} is the critical discount factor, and the properties of $f(\bar{q})$ and $g(\bar{q})$ described above assure that δ^{S} exists and satisfies $\delta^{S}=f(\bar{q})=g(\bar{q})$ for some $\bar{q}$. Hence we have

$$\pi^m+\delta^S\pi^m = \pi^{m*}+\delta^S\overline{\pi}, \qquad \overline{\pi}+\delta^S\pi^m = \overline{\pi}^*+\delta^S\overline{\pi} \tag{7}$$

for some $\overline{q}$. In the next section we compute $\overline{q}$ and δ^S satisfying (7).

3. Critical Discount Factors

In this section we compute the critical discount factor at which the maximal collusion is the outcome of some subgame perfect stick-and-carrot strategy profile. To this end, we rewrite (7) as

$$\delta^S = \frac{\pi^{m*}-\pi^m}{\pi^m-\overline{\pi}} = \frac{\overline{\pi}^*-\overline{\pi}}{\pi^m-\overline{\pi}}. \tag{8}$$

The latter equality in (8) implies

$$\pi^{m*}-\pi^m = \overline{\pi}^*-\overline{\pi}, \tag{9}$$

and it follows from (3) and (4) that

$$\pi^{m*}-\pi^m = \frac{(a-c)^2b^2(n-1)^2}{16(bn+e)^2(b+e)}. \tag{10}$$

As for the right-hand side of (9), we first assume that $\overline{\pi}^*>0$. Then $q^*((n-1)\overline{q})>0$; hence it follows from (1) that

$$\overline{q}^* \equiv q^*((n-1)\overline{q}) = \frac{(a-c)-b(n-1)\overline{q}}{2(b+e)},$$

$$\overline{\pi}^* = (b+e)(\overline{q}^*)^2 = \frac{[(a-c)-b(n-1)\overline{q}]^2}{4(b+e)}.$$

Routine computations yield

$$\overline{\pi}^*-\overline{\pi} = \frac{[(a-c)-b(n-1)\overline{q}]^2}{4(b+e)}-[(a-c)\overline{q}-(bn+e)\overline{q}^2]$$

$$= \frac{[[b(n+1)+2e]\overline{q}-(a-c)]^2}{4(b+e)}. \tag{11}$$

Substituting (10) and (11) into (9) and solving it, we obtain two values of $\overline{q}$, and the larger one is

$$\overline{q} = \frac{(a-c)[b(3n-1)+2e]}{2(bn+e)[b(n+1)+2e]}. \tag{12}$$

Hence

$$\bar{\pi} = \frac{(a-c)^2[b(3n-1)+2e][b(3-n)+2e]}{4(bn+e)[b(n+1)+2e]^2}. \tag{13}$$

This and (3) yield

$$\pi^m - \bar{\pi} = \frac{(a-c)^2 b^2 (n-1)^2}{(bn+e)[b(n+1)+2e]^2}. \tag{14}$$

Substituting (10) and (14) into (8), we obtain

$$\delta^S = \frac{\pi^{m*}-\pi^m}{\pi^m-\bar{\pi}} = \frac{[b(n+1)+2e]^2}{16(bn+e)(b+e)}. \tag{15}$$

Now, since we have assumed $\bar{\pi}^*>0$, we have $\bar{q}^*>0$. Hence it follows from (1) that $(a-c)-b(n-1)\bar{q}>0$. Substituting (12) into this inequality, we obtain

$$n-1 < 2(1+\sqrt{2})(1+e/b). \tag{16}$$

Also, it can be checked that if (16) holds then δ^S in (15) satisfies $\delta^S \in (0,1)$.

We next assume that $\bar{\pi}^*=0$. Then from (9) we have

$$\pi^m - \bar{\pi} = \pi^{m*}. \tag{17}$$

Substituting this into (8) and using (4) and (10), we have

$$\delta^S = \frac{\pi^{m*}-\pi^m}{\pi^m-\bar{\pi}} = \frac{\pi^{m*}-\pi^m}{\pi^{m*}} = \frac{b^2(n-1)^2}{[b(n+1)+2e]^2}. \tag{18}$$

Clearly, $\delta^S \in (0,1)$. It also follows from (17) and (10) that

$$\bar{\pi} = \pi^m - \pi^{m*} = -\frac{(a-c)^2 b^2 (n-1)^2}{16(bn+e)^2(b+e)}. \tag{19}$$

Putting $\bar{\pi}=(a-c)\bar{q}-(bn+e)\bar{q}^2$ in (19) and solving it, we obtain

$$\bar{q} = \frac{(a-c)[2(bn+e)(b+e)+[b(n+1)+2e]\sqrt{(bn+e)(b+e)}]}{4(bn+e)^2(b+e)}. \tag{20}$$

Since we have assumed $\bar{\pi}^*=0$, we have $\bar{q}^*=0$; hence (1) implies that $(a-c)-b(n-1)\bar{q}\leq 0$. Substituting (20) into this inequality yields

$$n-1 \geq 2(1+\sqrt{2})(1+e/b). \tag{21}$$

Summarizing, we have shown that if $\bar{\pi}^*>0$ then (15) and (16) hold true. On the other hand, if $\bar{\pi}^*=0$ then (18) and (21) hold true. Since (16) holds if and only if (21) does not hold, we have established:

Proposition. Let $\hat{n}$ be the smallest integer not less than $2(1+\sqrt{2})(1+e/b)$. Then

$$\delta^S = \begin{cases} \dfrac{[b(n+1)+2e]^2}{16(bn+e)(b+e)} & \text{if } n \leq \hat{n} \\[2ex] \dfrac{b^2(n-1)^2}{[b(n+1)+2e]^2} & \text{if } n > \hat{n}. \end{cases}$$

The following corollary is immediate from the above proposition.

Corollary. If $e=0$ (i.e., if marginal cost is constant), then

$$\delta^S = \begin{cases} \dfrac{(n+1)^2}{16n} & \text{if } n \leq 5 \\[2ex] \dfrac{(n-1)^2}{(n+1)^2} & \text{if } n \geq 6. \end{cases}$$

The case $n=2$ of the above corollary is obtained by putting $\gamma=1$ (no product differentiation) in the Cournot-type model of Lambertini and Sasaki (1999). This duopoly case has also been derived in Belleflamme and Peitz (2015, pp. 365–366).

As was mentioned in the Introduction, the results in Abreu (1986) imply that the critical discount factor δ^S in the above proposition is in fact the smallest common discount factor at which the maximal collusion is the outcome of some subgame perfect simple strategy profile with symmetric punishments. (Here, a punishment is said to be *symmetric* if all firms choose the same output at each period of the punishment path.) This result is derived as follows: Let δ' be the smallest common discount factor at which the maximal collusion is the outcome of some subgame perfect simple strategy profile with symmetric punishments. Then these punishments are the most severe symmetric punishments; for otherwise, δ' is not the smallest common discount factor. Since the Cournot-Nash equilibrium profit π^c in (2) is strictly positive, Theorems 13 and 14 in Abreu (1986) then guarantee that this subgame perfect simple strategy profile has a stick-and-carrot punishment structure. This implies that $\delta'=\delta^S$.[6)]

In the case of trigger strategies, we can compute from (2), (3), (4) and (5) that, for all n,

$$\delta^T = \frac{[b(n+1)+2e]^2}{[b(n+1)+2e]^2+4(bn+e)(b+e)}.$$

We can confirm that $\delta^S<\delta^T$ for all n. Also, if $e=0$, we have

$$\delta^T = \frac{(n+1)^2}{(n+1)^2+4n}.$$

The values of δ^S and δ^T for the cases $e=0$ and $e=b$ are as follows:

	$e=0$		$e=b$	
n	δ^S	δ^T	δ^S	δ^T
2	0.28	0.53	0.26	0.51
3	0.33	0.57	0.28	0.53
4	0.39	0.61	0.31	0.55
5	0.45	0.64	0.33	0.57
6	0.51	0.67	0.36	0.59
7	0.56	0.70	0.39	0.61
8	0.61	0.72	0.42	0.63
9	0.64	0.74	0.45	0.64

Compared with the case $e=0$, the values of δ^S and δ^T in the case $e=b$ are smaller. This is because, in the latter case, larger outputs incur higher marginal costs and hence deviations from the maximal collusion are less profitable.

The inverse demand function $F(Q)=a-bQ$ for all $Q\in[0,\infty)$ implies that the price $F(Q)$ is negative when $Q>a/b$; and in the present context a negative price is meaningless. It is worth noting, however, that the results in this paper hold true even if we assume instead that $F(Q)=\max\{a-bQ, 0\}$ for all $Q\in[0,\infty)$, provided that $c/a\geq 1/n$. In this case, there are three possibilities: (a) $\bar{\pi}^*>0$ and $F(n\bar{q})>0$; (b) $\bar{\pi}^*>0$ and $F(n\bar{q})=0$; (c) $\bar{\pi}^*=0$. When $c/a\geq 1/n$, case (b) never occurs. Also, it readily follows that (15) and (16) hold true for case (a), and that (18) holds true for case (c). Further, it can be shown that the assumptions $\bar{\pi}^*=0$ and $c/a\geq 1/n$ imply (21). Hence all the results continue to hold true for the case $F(Q)=\max\{a-bQ, 0\}$ for all $Q\in[0,\infty)$, provided that $c/a\geq 1/n$.[7)]

Finally, we note that the critical discount factor δ^S in the above proposition is the smallest common discount factor at which the maximal collusion is the outcome of some subgame perfect equilibrium in $G^\infty(\delta)$, regardless of whether the punishments are symmetric or asymmetric, *if and only if* $n>\hat{n}$. To see this, assume first that $n\leq\hat{n}$. Then we have $\bar{\pi}^*>0$; hence (7) implies that

$$\bar{\pi}+\frac{\delta^S\pi^m}{1-\delta^S}>0.$$

This means that the stick-and-carrot punishment yields strictly positive discounted profits. In this case, Theorem 35 in Abreu (1986) assures the existence of a subgame perfect simple strategy profile in $G^\infty(\delta^S)$ with more severe (asymmetric) punishments than the stick-and-carrot punishment. From this follows that the maximal collusion is the outcome of some subgame perfect equilibrium in $G^\infty(\delta^S-\varepsilon)$ for small enough $\varepsilon>0$. Hence δ^S is *not* the smallest common discount factor.

Assume next that $n > \hat{n}$. Then we have $\bar{\pi}^* = 0$, and (7) implies that

$$\bar{\pi} + \frac{\delta^S \pi^m}{1-\delta^S} = 0.$$

This means that the stick-and-carrot punishment is the most severe punishment. It therefore follows that δ^S is the smallest common discount factor.[8)]

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Notes

1) Abreu (1986) assumed that the cost function is linear, but the results referred to here also hold for quadratic cost functions with increasing marginal costs.
2) For details of infinitely repeated games with discounting, see Abreu (1988).
3) See, e.g., Belleflamme and Peitz (2015, p. 360).
4) Note that it suffices to check one-shot deviations for subgame perfection. The conditions are stated as Lemma 17 in Abreu (1986).
5) A proof that $g(\bar{q})$ is strictly increasing for all $\bar{q} > q^c$ is straightforward but lengthy; see Ushio (2019, Section 7).
6) After the pioneering work of Abreu (1986, 1988), the properties of subgame perfect equilibria with most severe punishments have been analyzed for various oligopoly games; e.g., Bertrand games (Lambson (1987, 1991)), Cournot-type and Bertrand-type games with product differentiation (Wernerfelt (1989), Lambertini and Sasaki (1999, 2002), Østerdal (2003)), Hotelling-type spatial competition games (Häckner (1996)).
7) See Ushio (2019, Section 5) for details.
8) The "if" part of the above result also follows from Theorem 1 in Segerstrom (1991). This theorem implies that the result holds when
$$\pi^{m*} - \pi^m \geq \pi^*((a-c)/b) - \pi((a-c)/b(n-1), (a-c)/b).$$
It can be verified that this inequality holds if $n > \hat{n}$.

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