On convexity of the aggregate production set

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Abstract

In this paper, we modify the C-antichain-convexity proposed by Ceparano and Quartieri (2019) and give a sufficient condition that leads to the convexity of the aggregate production set in the presence of the non-convexity of individual production technology. Our result may cover more general non-convexity on individual production sets.

1 Introduction

Convexity plays a crucial role in establishing the second welfare theorem. This theorem states that any Pareto efficient allocation can be supported as an equilibrium after proper redistribution of wealths[1)] For the proof of the theorem, it is standard to assume the convexity of individual production sets as well as the convexity of individual consumer's preferences to apply the separation theorem. However, the convexity of the aggregate production set is sufficient for the second welfare theorem as Debreu (1954) showed.

It is important to ask the question of what conditions on individual production sets lead to the convexity of the aggregate production set in the presence of the non-convexity of individual production technology? To answer this question Ceparano and Quartieri (2019) proposed notions of the C-antichain-convexity and the C-upwardness to extend the second welfare theorem in Debreu (1954). While the usual notion of convexity imposes the condition that the set includes a convex combination of two vectors included in the set, the C-antichain-convexity requires that the set should include a convex combination of any two vectors whose differences are not included in a fixed cone C. The C-upwardness is a kind of generalized notion of free-disposal in the sense that the set includes the sum of any vector in the set and a vector in a fixed cone C.

Using these notions, Ceparano and Quartieri (2019) showed that the sum of finitely many C-antichain-convex sets is convex if at least one of them is C-upward. Their results cover some kind of non-convex production sets having a finite number of kinks or stair-like shapes. However, they do not cover more general non-convex production sets.

In this paper, we modify the C-antichain-convexity and give a sufficient condition that leads to the convexity of the aggregate production set. Our result may cover more general non-convexity. On the other hand, we need to assume the free disposal condition on individual production sets. Moreover, although in Ceparano and Quartieri (2019) the commodity space is a real vector space, we confine it to a Euclidean space.

The organization of the paper is as follows: Section 2 presents Ceparano and Quartieri's (2019) mathematical results. In Section 3 we present the model and our result on the convexity of the aggregate production set. Section 4 provides the proof. Finally, we make some remarks on further research in Section 5.

2 Review of Ceparano and Quartieri (2019)'s results

In this section we present Ceparano and Quartieri (2019)'s mathematical results.

Let X and $X_1, X_2, \cdots, X_m$ be subsets in R^n. We also denote by C a cone in R^n.

The most fundamental concepts are C-chain-convexity, C-antichain-convexity, and C-upwardness in the following definitions.

Definition 2. 1 (C-chain-convexity and C-antichain-convexity)

- The set X is called <u>C-chain-convex</u> if, for all $x', x'' \in X$ with $x''-x' \in C$ and $\lambda \in [0, 1]$,

$$\lambda x' + (1-\lambda) x'' \in X.$$

- The set X is called <u>C-antichain-convex</u> if, for all $x', x'' \in X$ with $x''-x' \notin C \cup (-C)$ and $\lambda \in [0, 1]$,

$$\lambda x' + (1-\lambda) x'' \in X.$$

Definition 2. 2 (C-upwardness)

The set X is called <u>C-upward</u> if $x \in X$, $y \in V$ and $y-x \in C$ imply $y \in X$.

Definition 2. 3 (decomposable C-antichain-convexity)

The set X is called <u>decomposably C-antichain-convex</u> if X can be expressed as the sum of finitely many C-antichain-convex subsets of R^n.

Ceparano and Quartieri (2019) derived some results concerning the set satisfying C-antichain-convexity and C-upwardness which will be referred to in the later section.

Theorem 2. 4 (Ceparano and Quartieri (2019), Proposition 2)

The set X is convex if and only if X is C-chain-convex and C-antichain-convex.

Theorem 2. 5 (Ceparano and Quartieri (2019), Proposition 5)

The set X is C-chain-convex if X is C-upward.

Theorem 2. 6 (Ceparano and Quartieri (2019), Lemma 5)

The set X is C-upward if and only if $X+C\subset X$. Moreover, if $0\in C$, then, the set X is C-upward if and only if $X+C=X$.

Theorem 2. 7 (Ceparano and Quartieri (2019), Proposition 7)

If X_1 is C-upward, then X_1+X_2 is C-upward.

Finally, Ceparano and Quartieri (2019) derived a sufficient condition that the sum of finitely many subsets of R^n is convex, which is summarized in the following.

Theorem 2. 8 (Ceparano and Quartieri (2019), Corollary 1)

Suppose that $X_1, X_2, \cdots, X_m$ are C-antichain-convex. Moreover suppose that X_1 is C-upward. Then $X_1+X_2+\cdots+X_m$ is convex and C-upward.

This states that the sum of finitely many C-antichain-convex sets is convex if at least one of them is C-upward. Therefore, to check whether the sum of finitely many C-antichain-convex sets is convex or not, it suffices to show that (a) at least one of those is C-upward; and (b) the rest of these sets is C-antichain-convex.

3 Model and Result

In this section we present the main result on the convexity of the aggregate production set. Let us denote by $Y_i\subset R^n$ $(i=1, 2, \cdots, n)$ the individual production set. We also define the aggregate production set Y as the sum of individual production sets, i., e.,

$$Y \equiv \sum_{i=1}^{n} Y_i.$$

Let us also denote by $C\subset R^n$ a cone in R^n satisfying $0\in C$.

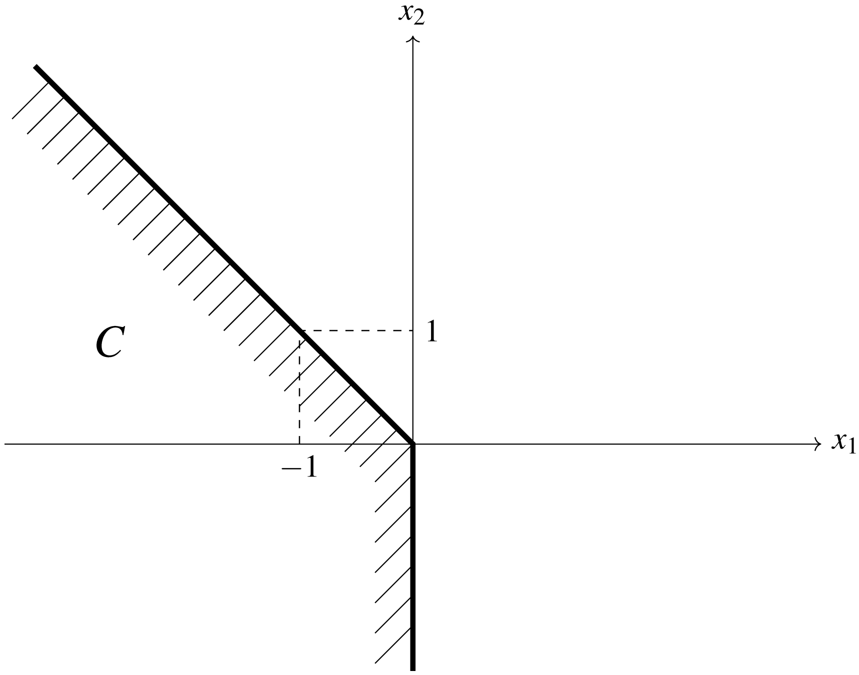

Figure 1: The set C

To extend the result on the convexity of the aggregate production set by Ceparano and Quartieri (2019), we need the following modified C-antichain-convexity in addition to free disposal property of individual production set.

Definition 3.1 (modified C-antichain-convexity) The set X is said to satisfy the modified C-antichain-convexity if X satisfies the following property:

for all $x', x'' \in X$ such that $x'' - x' \notin C \cup (-C)$, if there is $\hat{\lambda} \in [0,1]$ such that $\hat{\lambda} x' + (1-\hat{\lambda}) x'' \notin X$, then there are $\tilde{x}', \tilde{x}'' \in X$ with $\tilde{x}'' - \tilde{x}' \in C \cup (-C)$, and $\tilde{\lambda} \in [0,1]$ such that $\hat{\lambda} x' + (1-\hat{\lambda}) x'' \leqq \tilde{\lambda} \tilde{x}' + (1-\tilde{\lambda}) \tilde{x}''$.

Let us illustrate the modified C-antichain-convexity under free disposal. Figure 1 depicts the cone C in R^2. Given C, the set X is illustrated in Figure 2. Note that the set X dose not satisfy the condition of the C-antichain-convexity; moreover it is not decomposably C-antichain-convex. Thus, Theorem 2.8 (Ceparano and Quartieri (2019), Corollary 1) can not be applied. However, it is easily verified that it satisfies the modified C-antichain-convexity.

We now present the main result on the convexity of the aggregate production set under the modified C-antichain-convexity, which is the extension of Theorem 2.8 (Ceparano and Quartieri (2019), Corollary 1). We prove this in the following section.

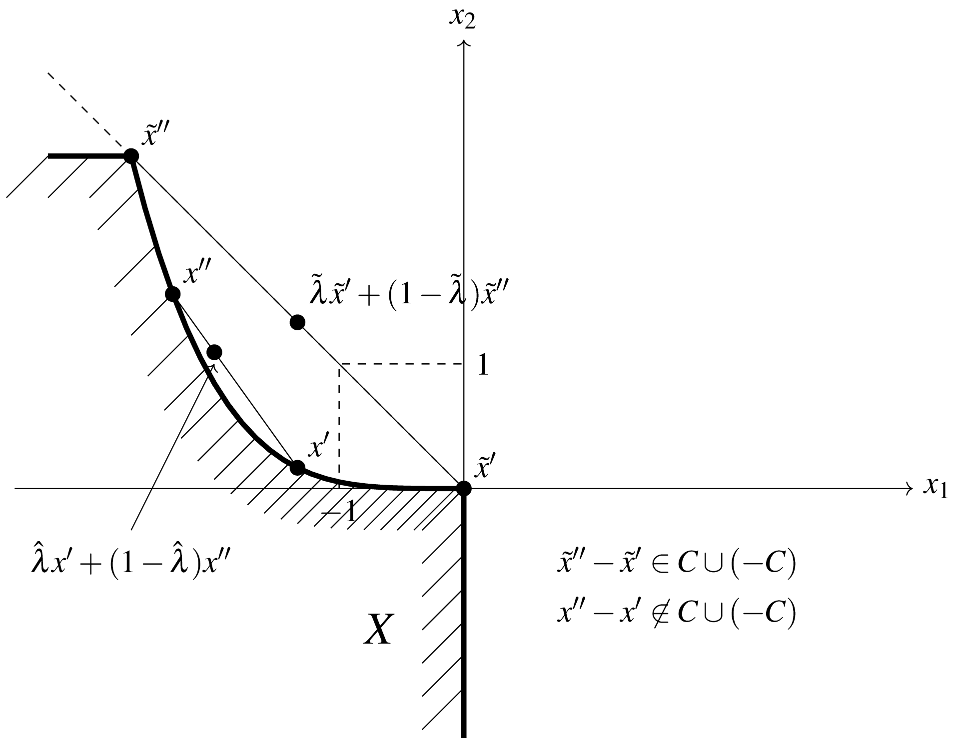

Figure 2: The set X satisfying the modified C-antichain-convexity

Theorem 3. 2　Let C and Y_i for all $i=1, 2, \cdots, n$ be subsets in R^n. Assume that:

(i) The sets Y_i for all $i=1, 2, \cdots, n$ satisfy the free disposal property;
　i.e., $Y_i - R^n_+ \subset Y_i$ for all $i=1, 2, \cdots, n$.

(ii) The set C is a convex cone with $0 \in C$.

(iii) The set Y_1 is C-upward and C-antichain-convex

(iv) For each $i=2, \cdots, n$, Y_i satisfies the modified C-antichain-convexity.

Then, $Y=\sum_{i=1}^{n} Y_i$ is convex and C-upward.

4　Proof of Theorem 3. 2

If we prove that Y_1+Y_2 is convex and C-upward, then by repeating the argument, we obtain the result that $Y=\sum_{i=1}^{n} Y_i$ is convex and C-upward.

First, we claim that Y_1 is convex. Since the assumption (iii) and Theorem 2. 5 (Ceparano and Quartieri (2019), Proposition 5) imply that Y_1 is C-chain-convex. Therefore, we conclude from Theorem 2. 4 (Ceparano and Quartieri (2019), Proposition 2) that Y_1 is convex.

It follows directly from Theorem 2.7 (Ceparano and Quartieri (2019), Proposition 7) that Y_1+Y_2 is C-upward. Note also from Theorem 2.6 (Ceparano and Quartieri (2019), Lemma 5) that C-upwardness of Y_1 is equivalent to the condition that $Y_1+C=Y_1$ since we assume that $0\in C$.

To show that Y_1+Y_2 is convex, let $z', z''\in Y_1+Y_2$ and $\lambda\in[0,1]$. Then, there exist $y_1', y_1''\in Y_1$ and $y_2', y_2''\in Y_2 \ni z'=y_1'+y_2'$ and $z''=y_1''+y_2''$. Since Y_1 is convex, we have

$$\lambda y_1'+(1-\lambda)y_1''+y_2' \in Y_1+Y_2, \tag{1}$$
$$\lambda y_1'+(1-\lambda)y_1''+y_2'' \in Y_1+Y_2. \tag{2}$$

Since $R^n=C\cup(-C)\cup\overline{(C\cup(-C))}$, we divide the remaining of the proof into 3 cases[2]:

case 1: $y_2''-y_2'\notin C\cup(-C)$;

case 2: $y_2''-y_2'\in C$;

case 3: $y_2''-y_2'\in(-C)$.

case 1: $y_2''-y_2'\notin C\cup(-C)$

If $\lambda y_2'+(1-\lambda)y_2''\in Y_2$, then the proof is done. Now suppose $\hat\lambda y_2'+(1-\hat\lambda)y_2''\notin Y_2$ for some $\hat\lambda$. It follows from the assumption (iv) that there are $\tilde y_2', \tilde y_2''\in Y_2$ with $\tilde y_2''-\tilde y_2'\in C\cup(-C)$, and $\tilde\lambda\in[0,1]$ such that

$$\hat\lambda y_2'+(1-\hat\lambda)y_2'' \leqq \tilde\lambda\tilde y_2'+(1-\tilde\lambda)\tilde y_2''.$$

Without loss of generality we assume that $\tilde y_2''-\tilde y_2' \in C$. Now, we claim that

$$\hat\lambda y_1'+(1-\hat\lambda)y_1''+\tilde\lambda\tilde y_2'+(1-\tilde\lambda)\tilde y_2'' \in Y_1+Y_2+C = Y_1+Y_2.$$

If this is true, then it follows from the free disposal property that

$$\hat\lambda z'+(1-\hat\lambda)z'' = \hat\lambda y_1'+(1-\hat\lambda)y_1''+\hat\lambda y_2'+(1-\hat\lambda)y_2'' \in Y_1+Y_2.$$

Note that $\hat\lambda y_1'+(1-\hat\lambda)y_1''\in Y_1$ and $\tilde y_2'\in Y_2$. From the fact that C is a cone and $\tilde y_2''-\tilde y_2'\in C$, it also follows that

$$(1-\tilde\lambda)(\tilde y_2''-\tilde y_2') \in C.$$

Therefore, we conclude that

$$\begin{aligned}&\hat\lambda y_1'+(1-\hat\lambda)y_1''+\tilde\lambda\tilde y_2'+(1-\tilde\lambda)\tilde y_2'' \\ &\quad = \{\hat\lambda y_1'+(1-\hat\lambda)y_1''\}+\tilde y_2'-(1-\tilde\lambda)\tilde y_2'+(1-\tilde\lambda)\tilde y_2''\end{aligned}$$

$$= \underbrace{\hat{\lambda} y_1' + (1-\hat{\lambda}) y_1'' + \tilde{y}_2'}_{\in Y_1+Y_2} + \underbrace{(1-\tilde{\lambda})(\tilde{y}_2'' - \tilde{y}_2')}_{\in C} \in Y_1 + Y_2 + C = Y_1 + Y_2.$$

case 2: $y_2'' - y_2' \in C$

Since C is a cone, it follows that $(1-\lambda)(y_2'' - y_2') \in C$. Therefore, we have the following relationship:

$$\begin{aligned} &\lambda y_1' + (1-\lambda) y_1'' + \lambda y_2' + (1-\lambda) y_2'' \\ &= \underbrace{\lambda y_1' + (1-\lambda) y_1'' + y_2'}_{\in Y_1+Y_2 \, by \, (1)} + \underbrace{(1-\lambda)(y_2'' - y_2')}_{\in C} \in Y_1 + Y_2 + C = Y_1 + Y_2. \end{aligned}$$

case 3: $y_2'' - y_2' \in (-C)$

Note that $y_2'' - y_2' \in (-C)$ is equivalent to $y_2' - y_2'' \in C$, which implies $(1-\lambda)(y_2'' - y_2') \in C$ because C is a cone. Thus, we have

$$\begin{aligned} &\lambda y_1' + (1-\lambda) y_1'' + \lambda y_2' + (1-\lambda) y_2'' \\ &= \underbrace{\lambda y_1' + (1-\lambda) y_1'' + y_2''}_{\in Y_1+Y_2 \, by \, (2)} + \underbrace{\lambda (y_2' - y_2'')}_{\in C} \in Y_1 + Y_2 + C = Y_1 + Y_2. \end{aligned}$$

This completes the proof. **Q. E. D.**

5 Concluding Remark

In this paper, we modify the C-antichain-convexity proposed by Ceparano and Quartieri (2019) and give a sufficient condition that leads to the convexity of the aggregate production set in the presence of the non-convexity of individual production technology. Our result may cover more general non-convexity on individual production sets.

However, our result is restrictive in several ways. First of all, while the commodity space of our framework is R^n, that of Ceparano and Quartieri (2019) is a real vector space. However, this extension may not be difficult. Second, the modified C-antichain-convexity heavily depends upon the free disposal property that is not assumed in Ceparano and Quartieri (2019). This is certainly restrictive and should be relaxed. Finally, our modified C-antichain-convexity is artificial and difficult to justify from a viewpoint of economics; therefore it should be replaced with some intuitive hypothesis.

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Notes

1) See, for example, Mas-Colell et. al. (1995).

2) Note that, if Y_2 is C-antichain-convex, then cases 2 and 3 are automatically satisfied.

References

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