On the comparative risk aversion of dynamically consistent stochastic differential utilities with the Epstein-Zin generator

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Abstract

This note is concerned with the comparative risk aversion property of stochastic differential utilities (SDUs) with the Epstein-Zin generator. The Epstein-Zin generator is highly non-linear and non-Lipschitz, and then we cannot use the usual Gronwall lemma to show their comparative risk aversion property. Accordingly, I here use the generalized Skiadas lemma instead of the usual Gronwall lemma. Applying the generalized Skiadas lemma, I show the comparative risk aversion property of various dynamically consistent SDUs including gain/loss asymmetric SDU and SDUs with Knightian uncertainty.

1. Introduction

In this note, I revisit the comparative risk aversion property of stochastic differential utilities (SDUs) with the Epstein-Zin generator. Duffie and Epstein (1992) show the comparative risk aversion property of SDUs with a Lipschitz driver. Therefore, the readers may think that it is natural that SDUs with the Epstein-Zin generator exhibit the comparative risk aversion property with respect to a coefficient of relative risk aversion, but, to the best of my knowledge, this rigorous proof has not been obtained yet. This is because the Epstein-Zin driver is highly non-linear and non-Lipschitz, so the standard Gronwall argument cannot be applied. However, in some cases, the Epstein-Zin driver exhibits the one-sided Lipschitz property, and then we can apply some type of the Gronwall argument to the Epstein-Zin SDUs. This note gives the proofs of the comparative risk aversion properties of various dynamically consistent Epstein-Zin SDUs under some parameter constraints to hold the one-sided Lipschitz property.

2. A Model and General Result

In this note, I only consider the Brownian uncertainty in the finite horizon. Let T be a fixed finitely positive real number and let K be a finite natural number. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space endowed with a K-dimensional Brownian motion $W := (W_t)_{t\in[0,T]}$. Further, let $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ be an augmented filtration generated by W. I say a stochastic process $C := (C_t)_{t\in[0,T]}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is a ***consumption process*** if it is right-continuous with left-limits and $\mathbb{F}$-progressively measurable, and it takes values in $(0, \infty)$. I here introduce a stochastic differential utility. Let f be a function from $(0, \infty)\times\mathbb{R}$ to $\mathbb{R}$. The function f is often referred to as a generator of the stochastic differential utility. Moreover, let u_T: $(0, \infty)\to\mathbb{R}$ be a function expressing the bequest utility. Then, I say a stochastic process $U := (U_t)_{t\in[0,T]}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is a ***stochastic differential utility*** (*SDU*) with (f, u_T) under a consumption process C if it satisfies the following for any $t\in[0, T]$:

$$U_t = \mathrm{E}_t\left[\int_t^T f(C_s, U_s)\mathrm{d}s + u_T(C_T)\right],$$

where E_t is a conditional expectation operator given $\mathcal{F}_t$. Additionally, I say an SDU U with (f, u_T) admits a backward stochastic differential equation (BSDE) representation on $(\Omega, \mathcal{F}, \mathbb{P})$ if for any consumption process C, there exists a right-continuous and $\mathbb{F}$-progressively measurable stochastic process $Z := (Z_t)_{t\in[0,T]}$ taking values in $\mathbb{R}^K$ such that

$$U_t = \int_t^T f(C_s, U_s)\mathrm{d}s - \int_t^T Z_s^\top \mathrm{d}W_s + u_T(C_T),$$

holds $\mathbb{P}$-almost surely for any $t\in[0, T]$, and U and Z satisfy

$$\mathrm{E}\left[\sup_{t\in[0,T]} |U_t|^2\right] < \infty \text{ and } \mathrm{E}\left[\int_0^T \|Z_s\|^2 \mathrm{d}s\right] < \infty,$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^K$.

In this note, I suppose the existence and uniqueness of U, but C should be constrained to satisfy conditions for the existence and uniqueness of U. Therefore, for any SDU U with (f, u_T), I introduce an admissible set of consumption processes, denoted by $\mathscr{C}_U$, such that for any $C\in\mathscr{C}_U$, U exists and is unique up to indistinguishability. However, in the Appendix, I provide the existence and uniqueness of the Epstein-Zin SDUs by using the theory of BSDEs. I here introduce a formal definition of comparative risk aversion following Duffie and Epstein (1992).

Definition 1: Comparative Risk Aversion

An SDU $\widetilde{U}$ is more risk-averse than another SDU U if they satisfy the following: for any deterministic consumption process $\overline{C}$ and stochastic consumption process C with $\overline{C}, C \in \mathscr{C}_U \cap \mathscr{C}_{\widetilde{U}}$, if $U_t(\overline{C}) \geq U_t(C)$ $\mathbb{P}$-almost surely for any $t \in [0, T]$, then it also holds $\widetilde{U}_t(\overline{C}) \geq \widetilde{U}_t(C)$ $\mathbb{P}$-almost surely for any $t \in [0, T]$.

Then, I have the following proposition, which is the main result of this note.

Proposition 2:

Suppose that two SDUs U with (f, u_T) and $\widetilde{U}$ with $(\tilde{f}, \tilde{u}_T)$ satisfy the following:

(1) U and $\widetilde{U}$ admit a BSDE representation on $(\Omega, \mathcal{F}, \mathbb{P})$.

(2) f satisfies the one-sided Lipschitz condition for v: there exists a constant $k \geq 0$ such that for any $c \in (0, \infty)$ and $v, w \in \mathbb{R}$,

$$(v-w)(f(c, v) - f(c, w)) \leq k(v-w)^2.$$

(3) There exists a twice-continuously differentiable function $h: \mathbb{R} \to \mathbb{R}$ that satisfies the following: there exists a constant $C_h \geq 0$ such that for any $c \in (0, \infty)$ and $v \in \mathbb{R}$, $h'(v) > 0$, $\frac{1}{h'(h^{-1}(v))} \leq C_h(1 + |v|)$, $h''(v) \leq 0$, $h(u_T(c)) = \tilde{u}_T(c)$, and the following equality holds.

$$f(c, v) = \frac{\tilde{f}(c, h(v))}{h'(v)}.$$

Then, $\widetilde{U}$ is more risk-averse than U.

To show Proposition 2, I need the following lemma obtained by Kraft et al. (2013).

Lemma 3: Theorem A. 2 in Kraft et al. (2013)

Suppose that a real-valued, right-continuous, and $\mathbb{F}$-progressively measurable process $X := (X_t)_{t \in [0, T]}$ satisfies $\mathrm{E}\left[\sup_{t \in [0, T]} |X_t|\right] < \infty$ with $X_T \geq 0$. Furthermore, there exists an $\mathbb{F}$-measurable process $A := (A_t)_{t \in [0, T]}$ and a constant $k \geq 0$ such that

$$X_t = \mathrm{E}_t\left[\int_t^T A_s \mathrm{d}s\right] \text{ for any } t \in [0, T],$$

and $A_t \geq kX_t$ on $\{X_t \leq 0\}$ for any $t \in [0, T]$. Then, $X_t \geq 0$ holds $\mathbb{P}$-almost surely for any $t \in [0, T]$.

Lemma 3 is a generalization of the Skiadas' lemma in Duffie and Epstein (1992). Here, let us show Proposition 2.

Proof. Let $\overline{C}$ and C be a deterministic consumption process and stochastic consumption process, respectively. Further, suppose that $\overline{C}, C \in \mathscr{C}_U \cap \mathscr{C}_{\tilde{U}}$. Note that by the condition (2), the inequality $f(c,v)-f(c,w) \geq k(v-w)$ holds for any $c \in (0,\infty)$ and $v, w \in \mathbb{R}$ with $v-w \leq 0$. By the BSDE representation property, I have

$$U_t(\overline{C}) = \int_t^T f(\overline{C}_s, U_s(\overline{C}))\mathrm{d}s + u_T(\overline{C}_T),$$

for any $t \in [0,T]$. Meanwhile, by the chain rule and the condition (3), I also have

$$\begin{aligned} h^{-1}(\tilde{U}_t(\overline{C})) &= \int_t^T (h^{-1})'(\tilde{U}_s(\overline{C}))\tilde{f}(\overline{C}_s, \tilde{U}_s(\overline{C}))\mathrm{d}s + h^{-1}(\tilde{u}_T(\overline{C}_T)) \\ &= \int_t^T \frac{\tilde{f}(\overline{C}_s, h(h^{-1}(\tilde{U}_s(\overline{C}))))}{h'(h^{-1}(\tilde{U}_s(\overline{C})))}\mathrm{d}s + u_T(\overline{C}_T) \\ &= \int_t^T f(\overline{C}_s, h^{-1}(\tilde{U}_s(\overline{C})))\mathrm{d}s + u_T(\overline{C}_T), \end{aligned}$$

for any $t \in [0,T]$. Hence, I have

$$U_t(\overline{C}) - h^{-1}(\tilde{U}_t(\overline{C})) = \int_t^T (f(\overline{C}_s, U_s(\overline{C})) - f(\overline{C}_s, h^{-1}(\tilde{U}_s(\overline{C}))))\mathrm{d}s,$$

for any $t \in [0,T]$. Then, if $U_t(\overline{C}) - h^{-1}(\tilde{U}_t(\overline{C})) \leq 0$ for some $t \in [0,T]$, by the condition (2), I have

$$f(\overline{C}_s, U_s(\overline{C})) - f(\overline{C}_s, h^{-1}(\tilde{U}_s(\overline{C}))) \geq k(U_t(\overline{C}) - h^{-1}(\tilde{U}_t(\overline{C}))).$$

Thus, Lemma 3 implies $U_t(\overline{C}) - h^{-1}(\tilde{U}_t(\overline{C})) \geq 0$. By applying the same argument to $h^{-1}(\tilde{U}_t(\overline{C})) - U_t(\overline{C})$, I obtain $h^{-1}(\tilde{U}_t(\overline{C})) - U_t(\overline{C}) \geq 0$, and thus $U_t(\overline{C}) = h^{-1}(\tilde{U}_t(\overline{C}))$.

By Ito's lemma, I have

$$\begin{aligned} h^{-1}(\tilde{U}_t(C)) &= \int_t^T (h^{-1})'(\tilde{U}_s(C))\tilde{f}(C_s, \tilde{U}_s(C))\mathrm{d}s - \frac{1}{2}\int_t^T (h^{-1})''(\tilde{U}_s(C))\|\tilde{Z}_s\|^2\mathrm{d}s \\ &\quad - \int_t^T (h^{-1})'(\tilde{U}_s(C))\tilde{Z}_s^\top \mathrm{d}W_s + h^{-1}(\tilde{u}_T(C_T)) \\ &= \int_t^T f(C_s, h^{-1}(\tilde{U}_s(C)))\mathrm{d}s + \frac{1}{2}\int_t^T \frac{h''(h^{-1}(\tilde{U}_s(C)))\|\tilde{Z}_s\|^2}{(h'(h^{-1}(\tilde{U}_s(C))))^3}\mathrm{d}s \\ &\quad - \int_t^T \frac{\tilde{Z}_s^\top \mathrm{d}W_s}{h'(h^{-1}(\tilde{U}_s(C)))} + u_T(C_T). \end{aligned}$$

Then, by the Burkholder-Davis-Gundy inequality, there exists a positive constant M such that

$$\mathrm{E}\left[\sup_{t\in[0,T]}\left|\int_0^t \frac{\tilde{Z}_s^\top \mathrm{d}W_s}{h'(h^{-1}(\tilde{U}_s(C)))}\right|\right]$$

$$\leq M\mathrm{E}\left[\sqrt{\int_0^T \frac{\|\tilde{Z}_s\|^2}{(h'(h^{-1}(\tilde{U}_s(C))))^2}\mathrm{d}s}\right]$$

$$\leq \frac{M}{2}\left(\mathrm{E}\left[\int_0^T \|\tilde{Z}_s\|^2\mathrm{d}s\right]+\mathrm{E}\left[\sup_{s\in[0,T]}\frac{1}{(h'(h^{-1}(\tilde{U}_s(C))))^2}\right]\right)$$

$$\leq \frac{M}{2}\left(\mathrm{E}\left[\int_0^T \|\tilde{Z}_s\|^2\mathrm{d}s\right]+2C_h^2+2C_h^2\mathrm{E}\left[\sup_{s\in[0,T]}|\tilde{U}_s(C)|^2\right]\right) < \infty.$$

Thus, $\left(\int_0^t \frac{\tilde{Z}_s{}^\top \mathrm{d}W_s}{h'(h^{-1}(\tilde{U}_s(C)))}\right)_{t\in[0,T]}$ is a true $\mathbb{F}$-martingale. Therefore, taking the conditional expectation of $h^{-1}(\tilde{U}_t(C))$, I have

$$h^{-1}(\tilde{U}_t(C)) = \mathrm{E}_t\left[\int_t^T\left\{f(C_s,h^{-1}(\tilde{U}_s(C)))+\frac{1}{2}Q(\tilde{U}_s(C))\|\tilde{Z}_s\|^2\right\}\mathrm{d}s+u_T(C_T)\right],$$

where $Q(v):=\frac{h''(h^{-1}(v))}{(h'(h^{-1}(v)))^3}\leq 0$. Therefore, I have

$$U_t(C)-h^{-1}(\tilde{U}_t(C)) = \mathrm{E}_t\left[\int_t^T\left\{f(C_s,U_s(C))-f(C_s,h^{-1}(\tilde{U}_s(C)))\right.\right.$$
$$\left.\left.-\frac{1}{2}Q(\tilde{U}_s(C))\|\tilde{Z}_s\|^2\right\}\mathrm{d}s\right],$$

for any $t\in[0,T]$. By the condition (2), there exists a constant $k\geq 0$ such that

$$f(C_s,U_s(C))-f(C_s,h^{-1}(\tilde{U}_s(C)))-\frac{1}{2}Q(\tilde{U}_s(C))\|\tilde{Z}_s\|^2 \geq k(U_s(C)-h^{-1}(\tilde{U}_s(C))),$$

if $U_s(C)-h^{-1}(\tilde{U}_s(C))\leq 0$. Thus, Lemma 3 implies $U_t(C)-h^{-1}(\tilde{U}_t(C))\geq 0$. Therefore, I obtain

$$h^{-1}(\tilde{U}_t(\bar{C})) = U_t(\bar{C}) \geq U_t(C) \geq h^{-1}(\tilde{U}_t(C)),$$

$\mathbb{P}$-almost surely for any $t\in[0,T]$. By the monotonicity of h, I obtain $\tilde{U}_t(\bar{C})\geq\tilde{U}_t(C)$. ***Q.E.D.***

In Proposition 2, I assume $\frac{1}{h'(h^{-1}(v))}\leq C_h(1+|v|)$. This condition is essential to satisfy the martingale condition of the stochastic integral term in Ito's formula, but this is strong because this condition is not satisfied when a coefficient of relative risk aversion is larger than one. However, this condition is needed because I compare U and $h^{-1}(\tilde{U})$. If I compare $h(U)$ and $\tilde{U}$, this condition can be replaced to $h'(v)\leq C_h(1+|v|)$. In fact, the above two cases cover almost all comparisons with respect to a coefficient of relative risk aversion. As in the proof of Proposition 2, I can show the following claim.

Corollary 4:

Suppose that two SDUs U with (f, u_T) and $\widetilde{U}$ with $(\tilde{f}, \tilde{u}_T)$ satisfy the following:

(1) U and $\widetilde{U}$ admit a BSDE representation on $(\Omega, \mathcal{F}, \mathbb{P})$.

(2) $\tilde{f}$ satisfies the one-sided Lipschitz condition for v: there exists a constant $k \geq 0$ such that for any $c \in (0, \infty)$ and $v, w \in \mathbb{R}$,

$$(v-w)(\tilde{f}(c, v) - \tilde{f}(c, w)) \leq k(v-w)^2.$$

(3) There exists a twice-continuously differentiable function $h: \mathbb{R} \to \mathbb{R}$ that satisfies the following: there exists a constant $C_h \geq 0$ such that for any $c \in (0, \infty)$ and $v \in \mathbb{R}$, $h'(v) > 0$, $h'(v) \leq C_h(1+|v|)$, $h''(v) \leq 0$, $h(u_T(c)) = \tilde{u}_T(c)$, and the following holds.

$$\tilde{f}(c, v) = h'(h^{-1}(v)) f(c, h^{-1}(v)).$$

Then, $\widetilde{U}$ is more risk-averse than U.

Proof. Let $\overline{C}$ and C be a deterministic consumption process and stochastic consumption process, respectively. Further, suppose that $\overline{C}, C \in \mathscr{C}_U \cap \mathscr{C}_{\widetilde{U}}$. Note that by condition (2), the inequality $\tilde{f}(c, v) - \tilde{f}(c, w) \geq k(v-w)$ holds for any $c \in (0, \infty)$ and $v, w \in \mathbb{R}$ with $v - w \leq 0$. By the BSDE representation property, I have

$$\widetilde{U}_t(\overline{C}) = \int_t^T \tilde{f}(\overline{C}_s, \widetilde{U}_s(\overline{C}))\mathrm{d}s + \tilde{u}_T(\overline{C}_T),$$

for any $t \in [0, T]$. Meanwhile, by the chain rule and the condition (3), I also have

$$\begin{aligned} h(U_t(\overline{C})) &= \int_t^T h'(U_s(\overline{C})) f(\overline{C}_s, U_s(\overline{C}))\mathrm{d}s + h(u_T(\overline{C}_T)) \\ &= \int_t^T h'(h^{-1}(h(U_s(\overline{C})))) f(\overline{C}_s, h^{-1}(h(U_s(\overline{C}))))\mathrm{d}s + \tilde{u}_T(\overline{C}_T) \\ &= \int_t^T \tilde{f}(\overline{C}_s, h(U_s(\overline{C})))\mathrm{d}s + \tilde{u}_T(\overline{C}_T), \end{aligned}$$

for any $t \in [0, T]$. Hence, I have

$$h(U_t(\overline{C})) - \widetilde{U}_t(\overline{C}) = \int_t^T (\tilde{f}(\overline{C}_s, h(U_s(\overline{C}))) - \tilde{f}(\overline{C}_s, \widetilde{U}_s(\overline{C})))\mathrm{d}s,$$

for any $t \in [0, T]$. Then, if $h(U_t(\overline{C})) - \widetilde{U}_t(\overline{C}) \leq 0$ for some $t \in [0, T]$, by the condition (2), I have

$$\tilde{f}(\overline{C}_s, h(U_s(\overline{C}))) - \tilde{f}(\overline{C}_s, \widetilde{U}_s(\overline{C})) \geq k(h(U_t(\overline{C})) - \widetilde{U}_t(\overline{C})).$$

Thus, Lemma 3 implies $h(U_t(\overline{C})) - \widetilde{U}_t(\overline{C}) \geq 0$. By applying the same argument to $\widetilde{U}_t(\overline{C}) - h(U_t(\overline{C}))$, I obtain $\widetilde{U}_t(\overline{C}) - h(U_t(\overline{C})) \geq 0$, and thus $h(U_t(\overline{C})) = \widetilde{U}_t(\overline{C})$.

By Ito's lemma, I have

$$h(U_t(C)) = \int_t^T h'(U_s(C))f(C_s, U_s(C))\mathrm{d}s - \frac{1}{2}\int_t^T h''(U_s(C))\|Z_s\|^2\mathrm{d}s$$

$$-\int_t^T h'(U_s(C))Z_s^\top \mathrm{d}W_s + h(u_T(C_T))$$

$$= \int_t^T \tilde{f}(C_s, h(U_s(C)))\mathrm{d}s - \frac{1}{2}\int_t^T h''(U_s(C))\|Z_s\|^2\mathrm{d}s$$

$$-\int_t^T h'(U_s(C))Z_s^\top \mathrm{d}W_s + \tilde{u}_T(C_T).$$

Then, by the Burkholder-Davis-Gundy inequality, there exists a positive constant M such that

$$\mathrm{E}\left[\sup_{t\in[0,T]}\left|\int_0^t h'(U_s(C))Z_s^\top \mathrm{d}W_s\right|\right] \leq M\mathrm{E}\left[\sqrt{\int_0^T \|Z_s\|^2(h'(U_s(C)))^2\mathrm{d}s}\right]$$

$$\leq \frac{M}{2}\left(\mathrm{E}\left[\int_0^T \|Z_s\|^2\mathrm{d}s\right] + \mathrm{E}\left[\sup_{s\in[0,T]}(h'(U_s(C)))^2\right]\right)$$

$$\leq \frac{M}{2}\left(\mathrm{E}\left[\int_0^T \|Z_s\|^2\mathrm{d}s\right] + 2C_h^2 + 2C_h^2\mathrm{E}\left[\sup_{s\in[0,T]}|U_s(C)|^2\right]\right) < \infty.$$

Thus, $\left(\int_0^t h'(U_s(C))Z_s^\top \mathrm{d}W_s\right)_{t\in[0,T]}$ is a true $\mathbb{F}$-martingale. Therefore, taking the conditional expectation of $h(U_t(C))$, I have

$$h(U_t(C)) = \mathrm{E}_t\left[\int_t^T \left\{\tilde{f}(C_s, h(U_s(C))) - \frac{1}{2}h''(U_s(C))\|Z_s\|^2\right\}\mathrm{d}s + \tilde{u}_T(C_T)\right].$$

Therefore, I have

$$h(U_t(C)) - \tilde{U}_t(C) = \mathrm{E}_t\left[\int_t^T \left\{\tilde{f}(C_s, h(U_s(C))) - \tilde{f}(C_s, \tilde{U}_s(C)) - \frac{1}{2}h''(U_s(C))\|Z_s\|^2\right\}\mathrm{d}s\right],$$

for any $t\in[0, T]$. By the condition (2), there exists a constant $k\geq 0$ such that

$$\tilde{f}(C_s, h(U_s(C))) - \tilde{f}(C_s, \tilde{U}_s(C)) - \frac{1}{2}h''(U_s(C))\|Z_s\|^2 \geq k(h(U_s(C)) - \tilde{U}_s(C)),$$

if $h(U_s(C)) - \tilde{U}_s(C) \leq 0$. Thus, Lemma 3 implies $h(U_t(C)) - \tilde{U}_t(C) \geq 0$. Therefore, by the monotonicity of h, I obtain

$$\tilde{U}_t(\overline{C}) = h(U_t(\overline{C})) \geq h(U_t(C)) \geq \tilde{U}_t(C),$$

$\mathbb{P}$-almost surely for any $t\in[0, T]$. Thus, I obtain $\tilde{U}_t(\overline{C}) \geq \tilde{U}_t(C)$. ***Q.E.D.***

3. Dynamically Consistent SDUs

I now consider specific SDUs. First, the original Epstein-Zin SDU by Duffie and Epstein (1992) with triplet (γ, ψ, δ) is defined as

$$f^{EZ}(c, v; \gamma, \psi, \delta) := \delta \frac{(1-\gamma)v}{1-\frac{1}{\psi}} \left\{ \frac{c^{1-\frac{1}{\psi}}}{((1-\gamma)v)^{\frac{1-\frac{1}{\psi}}{1-\gamma}}} - 1 \right\}, \qquad u_T^{EZ}(c; \gamma) := \frac{c^{1-\gamma}}{1-\gamma},$$

where $\gamma > 0$, $\gamma \neq 1$, $\psi > 0$, $\psi \neq 1$, and $\delta > 0$. δ expresses a subjective discount rate, γ represents a coefficient of relative risk aversion (RRA), and ψ is the elasticity of intertemporal substitution of consumption (EIS). Furthermore, the Epstein-Zin gain/loss asymmetric SDU, introduced by Shigeta (2020), with quadruplet $(\gamma, \psi, \underline{\delta}, \overline{\delta})$ is defined as

$$f^{GLA}(c, v; \gamma, \psi, \underline{\delta}, \overline{\delta}) := \min_{\delta \in [\underline{\delta}, \overline{\delta}]} \delta f^{EZ}(c, v; \gamma, \psi, 1), \qquad u_T^{GLA}(c; \gamma) := u_T^{EZ}(c; \gamma),$$

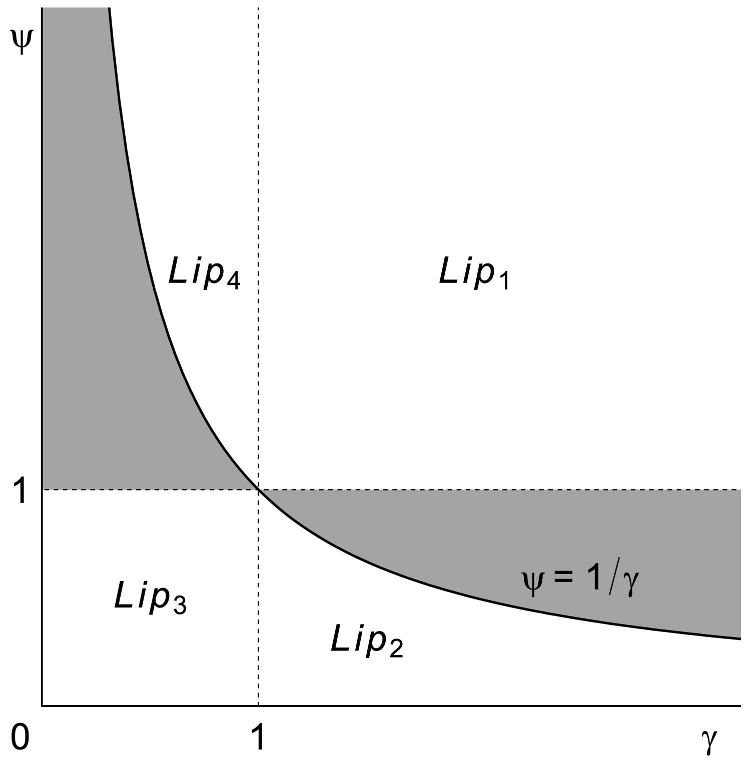
where $0 < \underline{\delta} \leq \overline{\delta} < \infty$. The gain/loss asymmetric SDU is an intertemporal version of loss averse utility. The comparative risk aversion properties of both of the SDUs are usually characterized by the parameter γ. Here, I have the following lemma for the Epstein-Zin generator.

Lemma 5: Proposition 3.2 in Kraft et al. (2013)
Suppose one of the following: (1) $\gamma > 1$ and $\psi > 1$, (2) $\gamma > 1$ and $\psi < 1$ with $\gamma\psi \leq 1$, (3) $\gamma < 1$ and $\psi < 1$, and (4) $\gamma < 1$ and $\psi > 1$ with $\gamma\psi \geq 1$. Then, the Epstein-Zin generator f^{EZ} and the Epstein-Zin gain/loss asymmetric generator f^{GLA} exhibit the one-sided Lipschitz property.

Thus, Proposition 2 and Corollary 4 can be applied in the four cases in Lemma 5. For any $i = 1, 2, 3$, and 4, I denote by Lip_i a set of (γ, ψ) which satisfies the condition (i) in Lemma 5. Figure 1 displays the four regions of (γ, ψ), Lip_i, $i = 1, 2, 3$, and 4. Then, I have the following proposition.

Proposition 6:
Fix $i = 1, 2, 3$, and 4. For any $\gamma, \gamma^*, \psi, \delta$ with $(\gamma, \psi), (\gamma^*, \psi) \in Lip_i$, the Epstein-Zin SDU with (γ^*, ψ, δ) is more risk-averse than the Epstein-Zin SDU with (γ, ψ, δ) if $\gamma \leq \gamma^*$ and both of the SDUs admit a BSDE representation on $(\Omega, \mathcal{F}, \mathbb{P})$. For any $\gamma, \gamma^*, \psi, \underline{\delta}, \overline{\delta}$ with $(\gamma, \psi), (\gamma^*, \psi) \in Lip_i$, the Epstein-Zin gain/loss asymmetric SDU with $(\gamma^*, \psi, \underline{\delta}, \overline{\delta})$ is more risk-averse than the Epstein-Zin gain/loss asymmetric SDU with $(\gamma, \psi, \underline{\delta}, \overline{\delta})$ if $\gamma \leq \gamma^*$ and

Figure 1. Regions of RRA γ and EIS ψ. Shadow regions are not considered in this note.



both of the SDUs admit a BSDE representation on $(\Omega, \mathcal{F}, \mathbb{P})$.

Proof. The conditions (1) and (2) in Proposition 2 and Corollary 4 are satisfied by the assumption and Lemma 5. So, I check the condition (3). Let

$$h(v;\gamma,\gamma^*) := \frac{1}{1-\gamma^*}((1-\gamma)v)^{\frac{1-\gamma^*}{1-\gamma}}.$$

Then, I have

$$h'(v;\gamma,\gamma^*) = ((1-\gamma)v)^{\frac{1-\gamma^*}{1-\gamma}-1} > 0,$$

$$h''(v;\gamma,\gamma^*) = (\gamma-\gamma^*)((1-\gamma)v)^{\frac{1-\gamma^*}{1-\gamma}-2} \leq 0,$$

$$h^{-1}(v;\gamma,\gamma^*) = \frac{1}{1-\gamma}((1-\gamma^*)v)^{\frac{1-\gamma}{1-\gamma^*}},$$

$$h(u_T^{EZ}(c;\gamma);\gamma,\gamma^*) = \frac{c^{1-\gamma^*}}{1-\gamma^*} = u_T^{EZ}(c;\gamma^*).$$

I first consider the case of $0<\gamma\leq\gamma^*<1$. In this case, I apply Proposition 2. Further, I suppose $2\gamma^*\leq 1+\gamma$. Then, I have

$$\frac{1}{h'(h^{-1}(v;\gamma,\gamma^*);\gamma,\gamma^*)}=((1-\gamma)h^{-1}(v;\gamma,\gamma^*))^{1-\frac{1-\gamma^*}{1-\gamma}}=((1-\gamma^*)v)^{\frac{1-\gamma}{1-\gamma^*}\left(1-\frac{1-\gamma^*}{1-\gamma}\right)}$$
$$=((1-\gamma^*)v)^{\frac{\gamma^*-\gamma}{1-\gamma^*}}.$$

Here, since $2\gamma^*\leq 1+\gamma$, the exponent of the above satisfies

$$0\leq\frac{\gamma^*-\gamma}{1-\gamma^*}\leq\frac{\frac{1+\gamma}{2}-\gamma}{1-\frac{1+\gamma}{2}}=1.$$

Thus, I have

$$\frac{1}{h'(h^{-1}(v;\gamma,\gamma^*);\gamma,\gamma^*)}\leq 1+|(1-\gamma^*)v|\leq 1+|v|.$$

Furthermore, I have

$$\frac{f^{EZ}(c,h(v);\gamma^*,\psi,\delta)}{h'(v)}=((1-\gamma)v)^{1-\frac{1-\gamma^*}{1-\gamma}}\delta\frac{((1-\gamma)v)^{\frac{1-\gamma^*}{1-\gamma}}}{1-\frac{1}{\psi}}\left\{\frac{c^{1-\frac{1}{\psi}}}{(((1-\gamma)v)^{\frac{1-\gamma^*}{1-\gamma}})^{\frac{1-\frac{1}{\psi}}{1-\gamma^*}}}-1\right\}$$
$$=\delta\frac{(1-\gamma)v}{1-\frac{1}{\psi}}\left\{\frac{c^{1-\frac{1}{\psi}}}{((1-\gamma)v)^{\frac{1-\frac{1}{\psi}}{1-\gamma}}}-1\right\}=f^{EZ}(c,v;\gamma,\psi,\delta).$$

Therefore, by Proposition 2, the Epstein-Zin SDU with (γ^*,ψ,δ) is more risk-averse than the Epstein-Zin SDU with (γ,ψ,δ). If $2\gamma^*>1+\gamma$, then let $\gamma_k=\frac{1+\gamma_{k-1}}{2}$ for $k\geq 1$ and $\gamma_0=\gamma$. Then, I have

$$\gamma_k=1-\frac{1-\gamma}{2^k},$$

for any $k\geq 0$. Furthermore, $\gamma_k\to 1$ as $k\to\infty$. Comparing the SDU with (γ_k,ψ,δ) and one with $(\gamma_{k-1},\psi,\delta)$ for any $k\geq 1$, I obtain the comparative risk aversion property with respect to γ_k and γ_{k-1} for any $k\geq 1$. Thus, by the induction, I obtain that the SDU with (γ_k,ψ,δ) is more risk-averse than the SDU with (γ,ψ,δ), for any $k\geq 1$. Let k^* be a natural number such that $\gamma_{k^*}<\gamma^*\leq\gamma_{k^*+1}$. Then, I have $2\gamma^*\leq\gamma_{k^*}+1$. Hence, by the comparison between γ^* and γ_{k^*}, I obtain that the SDU with (γ^*,ψ,δ) is more risk-averse than the SDU with $(\gamma_{k^*},\psi,\delta)$. Therefore, I finally obtain that the SDU with (γ^*,ψ,δ) is more risk-averse than the SDU with (γ,ψ,δ).

I next consider the case of $1<\gamma\leq\gamma^*$. In this case, I apply Corollary 4. Further, I suppose $2\gamma\geq 1+\gamma^*$. Then, since $0\leq\frac{\gamma^*-\gamma}{\gamma-1}\leq 1$, I have

$$h'(v;\gamma,\gamma^*) = ((1-\gamma)v)^{\frac{1-\gamma^*}{1-\gamma}-1} = ((1-\gamma)v)^{\frac{\gamma^*-\gamma}{\gamma-1}} \leq 1+|(1-\gamma)v|$$

$$\leq \max\{1,|1-\gamma|\}(1+|v|).$$

Furthermore, I have

$$h'(h^{-1}(v;\gamma,\gamma^*);\gamma,\gamma^*)f^{EZ}(c,h^{-1}(v;\gamma,\gamma^*);\gamma,\psi,\delta)$$

$$= ((1-\gamma^*)v)^{1-\frac{1-\gamma}{1-\gamma^*}}\delta\frac{((1-\gamma^*)v)^{\frac{1-\gamma}{1-\gamma^*}}}{1-\frac{1}{\psi}}\left\{\frac{c^{1-\frac{1}{\psi}}}{\left(((1-\gamma^*)v)^{\frac{1-\gamma}{1-\gamma^*}}\right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}}}-1\right\}$$

$$= \delta\frac{(1-\gamma^*)v}{1-\frac{1}{\psi}}\left\{\frac{c^{1-\frac{1}{\psi}}}{((1-\gamma^*)v)^{\frac{1-\frac{1}{\psi}}{1-\gamma^*}}}-1\right\} = f^{EZ}(c,v;\gamma^*,\psi,\delta).$$

Thus, by Corollary 4, I obtain that the SDU with (γ^*,ψ,δ) is more risk-averse than the SDU with (γ,ψ,δ). The assumption $2\gamma\geq 1+\gamma^*$ can be removed similarly to the case of $0<\gamma\leq\gamma^*<1$. The case of the gain/loss asymmetric SDU can be shown similarly. Note that the one-sided Lipschitz property of the gain/loss asymmetric SDU follows from the trivial inequality: $\min\{x\}-\min\{y\}\leq\max\{x-y\}$. ***Q.E.D.***

4. Dynamically Consistent SDUs with Knightian Uncertainty of the Drift

The literature proposes dynamically consistent SDUs with Knightian uncertainty with respect to the drift of the Brownian motion, such as Chen and Epstein (2002), and Beissner et al. (2020). These SDUs can be typically characterized as a solution to the following BSDE on $(\Omega,\mathcal{F},\mathbb{P})$.

$$U_t = \int_t^T \{f(C_s,U_s)+e(s,Z_s)\}\mathrm{d}s - \int_t^T Z_s^\top \mathrm{d}W_s + u_T(C_T).$$

The term $e(s,Z_s,\omega)$ expresses the adjustment of Knightian uncertainty with respect to the drift of W. In a case of the max-min SDU by Chen and Epstein (2002), e is expressed as

$$e(t,z) := \min_{\theta\in\Theta_t}\theta^\top z,$$

where $\Theta_t\subseteq\mathbb{R}^K$ is a correspondence that represents the region of considerable candidates of the uncertain drift vector of W. Without loss of generality, I suppose $\{0\}\in\Theta_t$ for all $t\in[0,T]$. In a case of the recursive α-maxmin SDU by Beissner et al. (2020), e is expressed as

$$e(t,z) := (1-\alpha_t)\left(\max_{\theta\in\Theta_t}\theta^\top z\right)+\alpha_t\left(\min_{\theta\in\Theta_t}\theta^\top z\right),$$

where $a_t \in [0, 1]$ represents the weight of optimism and pessimism. I call the above two SDUs an SDU with Knightian uncertainty. Further, I call SDUs with Knightian uncertainty whose generator f is the Epstein-Zin generator an Epstein-Zin SDU with Knightian uncertainty.

By the term e, Proposition 2 and Corollary 4 cannot be applied for general considerable consumption processes. However, in consumptions with $e=0$, I obtain the comparative risk aversion property, as discussed in Chen and Epstein (2002). Therefore, I define the unambiguous consumption process as follows.

Definition 7: Unambiguous Consumption Processes

Let U be an SDU with Knightian uncertainty admitting a BSDE representation on $(\Omega, \mathcal{F}, \mathbb{P})$. A consumption process $C \in \mathscr{C}_U$ is unambiguous if a corresponding solution to the BSDE, $(Z_t)_{t\in[0,T]} = (Z_t^1, \cdots, Z_t^K)^\top_{t\in[0,T]}$, and correspondences of priors, $(\Theta_t)_{t\in[0,T]}$, satisfy $\mathrm{proj}_i(\Theta_t) = \{0\}$ if $Z_t^i \neq 0$ for any $t \in [0, T]$ and $i = 1, \cdots, K$, where $\mathrm{proj}_i(A)$ is a projection mapping set of $A \subseteq \mathbb{R}^K$ to the i-th dimension of A.

Then, I have the following proposition about the comparative risk aversion property in unambiguous consumption processes.

Proposition 8:

Fix $i = 1, 2, 3$, and 4. For any $\gamma, \gamma^*, \psi, \delta$ with $(\gamma, \psi), (\gamma^*, \psi) \in Lip_i$, when consumption processes are restricted to be unambiguous, the Epstein-Zin SDU with (γ^*, ψ, δ) and Knightian uncertainty is more risk-averse than the Epstein-Zin SDU with (γ, ψ, δ) and Knightian uncertainty if $\gamma \leq \gamma^*$ and both of the SDUs admit a BSDE representation on $(\Omega, \mathcal{F}, \mathbb{P})$.

Proof. Since consumption processes are restricted to be unambiguous, I have $e(t, Z_t) = 0$, $\mathbb{P}$-almost surely for any $t \in [0, T]$. Thus, in this case, we can apply Proposition 2 and Corollary 4 to the Epstein-Zin SDU with Knightian uncertainty, and I obtain the desired result. ***Q.E.D.***

5. Concluding Remarks

This note considers the comparative risk aversion property of the Epstein-Zin utilities in continuous time. Focusing on the Brownian uncertainty, I show that the various classes of the Epstein-Zin utility exhibit the comparative risk aversion, under the assumption of the coefficient of relative risk aversion and the elasticity of intertemporal substitution. There are several potential extensions of this paper. The first is another type of uncertainty, such as the Poissonian uncertainty. In many applications, the Poisson process introduces interesting and realistic behaviors of the agent. The second is to remove the assumption about the values of the relative risk aversion and elasticity of intertemporal substitution. In particular, this note's results do not cover the case of $\gamma > 1$ and $\psi < 1$ with $\gamma\psi > 1$. This case is often assumed in macroeconomics. The third is a comparison of different cases in Lemma 5. It is not clear, for example, whether the agent who has the Epstein-Zin utility with $\gamma = 1.5$ and $\psi = 1.5$ is more risk-averse than another agent who has the Epstein-Zin utility with $\gamma = 0.8$ and $\psi = 1.5$. The above extensions are left for future research.

Appendix: The Existence and Uniqueness of a Solution to the BSDE for Dynamically Consistent Epstein-Zin SDUs

In this appendix, I discuss the existence and uniqueness of a solution to the BSDE for SDUs. If a consumption process is Markov, then the existence and uniqueness of an SDU are replaced with the existence and uniqueness of a (viscosity) solution to some PDE, like in Duffie and Lions (1992). This is a natural consequence of the Feynman-Kac representation theorem. However, this result heavily relies on the definition of the consumption process. Indeed, Markov consumption processes yielding a well-posed PDE are restricted. Furthermore, the restriction of the Markov consumption process is problematic to show the necessity of the Hamilton-Jacobi-Bellman equation approach for solving the utility maximization problem. Therefore, I consider a more general setting. I here use the existence and uniqueness result for a semilinear BSDE, obtained by Pardoux and Răşcanu (2014) in their Proposition 5.24. Shigeta (2022) employes the same approach to show the existence and uniqueness of the Epstein-Zin quasi-hyperbolic discounting SDU.

In general, this note considers an SDU that would be a solution to the following BSDE on $(\Omega, \mathcal{F}, \mathbb{P})$:

$$U_t = \int_t^T \{f(C_s, U_s) + e(t, Z_s)\} \mathrm{d}s - \int_t^T Z_s^\top \mathrm{d}W_s + u_T(C_T).$$

I here suppose the following condition for the adjusted term of Knightian uncertainty, e.

Assumption 9:

The adjusted term of Knightian uncertainty, e, satisfies the following.

(1) There exists a positive constant L such that for any $z, z' \in \mathbb{R}^K$,

$$|e(t,z)-e(t,z')| \leq L\|z-z'\|,$$

holds $\mathbb{P}$-almost surely for any $t \in [0, T]$.

(2) $e(t, 0)=0$ holds $\mathbb{P}$-almost surely for any $t \in [0, T]$.

(3) $(e(t, Z_t))_{t\in[0,T]}$ is $\mathbb{F}$-progressively measurable if $(Z_t)_{t\in[0,T]}$ is $\mathbb{F}$-progressively measurable as well.

By the conditions (1) and (2) in Assumption 9, $z \to e(t,z)$ satisfies a linear growth condition for any $t \in [0, T]$: $|e(t,z)| \leq L\|z\|$ for any $z \in \mathbb{R}^K$. Note that the models in section 4 satisfy Assumption 9 if the correspondence of priors, Θ_t, is deterministic and time independent. Hereafter, I only consider the cases in which the generator and the bequest utility satisfy $f(c,v)=f^{EZ}(c,v;\gamma,\psi,\delta)$ and $u_T(c)=u_T^{EZ}(c;\gamma)$. However, the following discussion can be applied in the cases of the other Epstein-Zin SDUs in this note, such as the gain/loss asymmetric Epstein-Zin SDU. Furthermore, I restrict consumption processes as follows.

Definite 10: Admissible Consumption

A consumption process $(C_t)_{t\in[0,T]}$ is admissible if it is right-continuous with left limits and $\mathbb{F}$-progressively measurable, and it satisfies

$$\mathrm{E}\left[\int_0^T C_t^{2\left(1-\frac{1}{\psi}\right)}\mathrm{d}t\right] < \infty, \qquad \text{and} \qquad \mathrm{E}[C_T^{2(1-\gamma)}] < \infty.$$

The admissibility of consumptions implies the square integrability of the generator and the terminal value of the BSDEs.

To apply Proposition 5.24 in Pardoux and Răşcanu (2014), let us change the variable from U to $X:=(1-\gamma)U-1$. Then, the BSDE of X is

$$X_t = \int_t^T \{f^{EZ,X}(C_s, X_s;\gamma,\psi,\delta)+e^X(t, Z_s^X)\}\mathrm{d}s - \int_t^T (Z_s^X)^\top \mathrm{d}W_s + C_T^{1-\gamma}+1,$$

where

$$f^{EZ,X}(c,x;\gamma,\psi,\delta) := \delta\theta\left(c^{1-\frac{1}{\psi}}(x+1)^{1-\frac{1}{\theta}}-(x+1)\right),$$

$$e^X(t,z) := (1-\gamma)e\left(t, \frac{z}{1-\gamma}\right), \qquad \theta := \frac{1-\gamma}{1-\frac{1}{\psi}}.$$

In the above representation, I use the symbol θ, which has been used to a prior in the Knightian uncertainty models. However, I hereafter do not use a prior, so I believe there is no confusion. For stability, I hereafter suppose $\theta<0$, so that x can take -1. Furthermore, I consider the following BSDE.

$$X_t = \int_t^T \{f_0^{EZ,X}(C_s, X_s; \gamma, \psi, \delta) + e^X(t, Z_s^X)\}\mathrm{d}s - \int_t^T (Z_s^X)^\top \mathrm{d}W_s + C_T^{1-\gamma} + 1,$$

where

$$f_0^{EZ,X}(c, x; \gamma, \psi, \delta) := \delta\theta(c^{1-\frac{1}{\psi}}((x+1)\vee 0)^{1-\frac{1}{\theta}} - (x+1)).$$

Then, x can take an arbitrary value in $\mathbb{R}$ since $\theta<0$. If there exists a solution to the BSDE with the driver $f_0^{EZ,X}$, we can easily see by Lemma 3 that this solution also solves the BSDE with the driver $f^{EZ,X}$ under the one-sided Lipschitz property. Therefore, I hereafter focus on the solvability of the BSDE of X with the driver $f_0^{EZ,X}$ under the assumptions of $\theta<0$. Note that $f_0^{EZ,X}$ also holds the one-sided Lipschitz property if $\theta<0$.

Here, I check the conditions for the existence and uniqueness results of Pardoux and Ră şcanu (2014), in the line of the setting of this note. The conditions in Pardoux and Răşcanu (2014), called (BSDE-MH 0_Φ), are as follows.

(a) T is finite.

(b) $C_T^{1-\gamma}+1$ is $\mathcal{F}_T$-measurable and square-integrable.

(c)

(i) For any $x\in\mathbb{R}$ and $z\in\mathbb{R}^K$, $(f_0^{EZ,X}(C_t, x; \gamma, \psi, \delta) + e^X(t, z))_{t\in[0,T]}$ is $\mathbb{F}$-progressively measurable.

(ii) For any constant $\rho>0$,

$$\int_0^T \sup_{|x|\leq\rho} |f_0^{EZ,X}(C_t, x; \gamma, \psi, \delta)|\mathrm{d}t < \infty,$$

$\mathbb{P}$-almost surely.

(iii) For any constant $\rho>0$,

$$\mathrm{E}\left[\int_0^T \sup_{|x|\leq\rho} |f_0^{EZ,X}(C_t, x; \gamma, \psi, \delta)|^2\mathrm{d}t\right] < \infty.$$

(iv) There exists a positive constant L such that for any $z, z'\in\mathbb{R}^K$,

$$|e^X(t, z) - e^X(t, z')| \leq L\|z - z'\|,$$

holds $\mathbb{P}$-almost surely for any $t\in[0, T]$.

(v) There exists a constant $k\geq 0$ such that for any $t\in[0, T]$ and $x, y\in\mathbb{R}$,

$$(x-y)(f_0^{EZ,X}(C_t, x; \gamma, \psi, \delta) - f_0^{EZ,X}(C_t, y; \gamma, \psi, \delta)) \leq k(x-y)^2.$$

(vi) $x\to f_0^{EZ,X}(C_t, x; \gamma, \psi, \delta)$ is continuous for any $t\in[0, T]$.

Under the one-sided Lipschitz property of $f_0^{EZ,X}$, $\theta<0$, Assumption 9, and the admissibility of the consumption $C=(C_t)_{t\in[0,T]}$, the conditions (a), (b), (c)-(i), (iv), (v), and (vi) are satisfied. For the condition (c)-(ii), I have the following inequality.

$$\int_0^T \sup_{|x|\leq\rho} |f_0^{EZ,X}(C_t, x; \gamma, \psi, \delta)| \mathrm{d}t \leq |\delta\theta| \left((1+\rho)^{1-\frac{1}{\theta}} \int_0^T C_t^{1-\frac{1}{\psi}} \mathrm{d}t + T(1+\rho) \right) < \infty,$$

$\mathbb{P}$-almost surely for any constant $\rho > 0$. This inequality is satisfied only when $\theta < 0$ or $\theta = 1$. The integrability of C implies the last inequality. Additionally, for the condition (c)-(iii), I have

$$\mathrm{E}\left[\int_0^T \sup_{|x|\leq\rho} |f_0^{EZ,X}(C_t, x; \gamma, \psi, \delta)|^2 \mathrm{d}t\right]$$

$$\leq 2|\delta\theta|^2 \left((1+\rho)^{2\left(1-\frac{1}{\theta}\right)} \mathrm{E}\left[\int_0^T C_t^{2\left(1-\frac{1}{\psi}\right)} \mathrm{d}t\right] + T(1+\rho)^2 \right) < \infty,$$

Where I have used the inequality $|x+y|^2 \leq 2(|x|^2 + |y|^2)$. Thus, all the conditions in (BSDE-MH 0_Φ) are satisfied. Hence, the solution of X uniquely exists. By the one-sided Lipschitz property we can show that $X+1$ is always non-negative. Therefore, we can remove the truncation $(x+1) \vee 0$, and obtain $U = (X+1)/(1-\gamma)$.

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