A Note on Convex Aggregate Production Set under Star-shapedness

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Abstract

Hamano (2022) modified the C-antichain-convexity proposed by Ceparano and Quartieri (2019) and provided a sufficient condition that leads to the convexity of the aggregate production set in the presence of non-convex individual production technology. However, Hamano (2022) assumed the free disposal condition, which is not required in Ceparano and Quartieri (2019). In this paper, we modify the assumption proposed by Hamano (2022) under free disposal and provide a sufficient condition that guarantees the convexity of the aggregate production set without assuming free disposal.

1 Introduction

The second welfare theorem states that any Pareto efficient allocation can be supported as a price equilibrium after proper redistribution of wealth[1]. To prove the theorem, the convexity of individual production sets as well as the convexity of the preferences of individual consumers is usually assumed in order to apply the separation theorem. However, as noted by Debreu (1954), the convexity of the aggregate production set is sufficient to establish the second welfare theorem.

It remains interesting and important to investigate the conditions on individual production sets to ensure the convexity of the aggregate production set in the presence of non-convex individual production technologies. Ceparano and Quartieri (2019) addressed this question and proposed notions of C-antichain convexity and C-upwardness, given a cone C, to extend the second welfare theorem presented by Debreu (1954). Using these new concepts, Ceparano and Quartieri (2019) proved that the sum of finitely many C-antichain-convex sets is convex if at least one of the sets is C-upward. Their findings encompass a "mild" degree of non-convexity, including certain types of non-convex production sets having a finite number of kinks or stair-like shapes. However, their results do not accommodate more general non-convex production sets. The usual notion of convexity imposes the requirement that a set includes a convex combination of any two vectors in the set. However, the C-antichain-convexity requires that the set must include a convex combination of any two vectors whose difference is not included in a fixed cone C. The C-upwardness is a kind of generalized notion of free disposal in the direction of C, in the sense that the set must include the sum of any vector in the set and a vector in the fixed cone C. Note that, if a set satisfies both C-antichain-convexity and C-upwardness, the set is convex.

Hamano (2022) modified the concept of C-antichain-convexity proposed by Ceparano and Quartieri (2019) and provided a sufficient condition for the convexity of the aggregate production set, even in the presence of non-convex individual production technologies. However, Hamano (2022) assumed the free disposal condition, which is not required in Ceparano and Quartieri (2019). In this paper, we modify the assumption proposed by Hamano (2022) under free disposal and provide a sufficient condition that leads to the convexity of the aggregate production set without assuming the free disposal condition for individual production sets.

The organization of the paper is as follows: Section 2 presents the mathematical results of Ceparano and Quartieri's (2019). The model and the main result concerning the convexity of the aggregate production set are presented in Section 3. Section 4 provides the proof.

2 Review of Ceparano and Quartieri (2019)'s results

In this section mathematical results in Ceparano and Quartieri (2019) are presented[2]. Let X and $X_1, X_2, \cdots, X_m$ be subsets in R^n. We also denote by C a cone in R^n.

Given the cone C, the most important concepts are C-chain-convexity, C-antichain-convexity, and C-upwardness in the following definitions.

Definition 2. 1 (C-chain-convexity and C-antichain-convexity)

- The set X is called <u>C-chain-convex</u> if, for all $x', x'' \in X$ with $x'' - x' \in C$ and $\lambda \in [0, 1]$,

$$\lambda x' + (1-\lambda) x'' \in X.$$

- The set X is called <u>C-antichain-convex</u> if, for all $x', x'' \in X$ with $x'' - x' \notin C \cup (-C)$ and $\lambda \in [0, 1]$,

$$\lambda x' + (1-\lambda) x'' \in X.$$

Definition 2. 2 (C-upwardness)

The set X is called C-upward if $x \in X$, $y \in R^n$ and $y - x \in C$ imply $y \in X$.

Ceparano and Quartieri (2019) obtained several propositions concerning sets that satisfy C-antichain-convexity and C-upwardness. These will be referred to in the later sections.

Proposition 2. 3 (Ceparano and Quartieri (2019, Proposition 2))

The set X is convex if and only if X is C-chain-convex and C-antichain-convex.

Proposition 2. 4 (Ceparano and Quartieri (2019, Proposition 5))

The set X is C-chain-convex if X is C-upward.

Proposition 2. 5 (Ceparano and Quartieri (2019, Lemma 5))

The set X is C-upward if and only if $X + C \subset X$. Moreover, if $0 \in C$, then, the set X is C-upward if and only if $X + C = X$.

Proposition 2. 6 (Ceparano and Quartieri (2019, Proposition 7))

If X_1 is C-upward, then $X_1 + X_2$ is C-upward.

Finally, a sufficient condition for the convexity of the sum of finitely many subsets of R^n is derived and summarized as follows:

Proposition 2. 7 (Ceparano and Quartieri (2019, Corollary 1))

Suppose that $X_1, X_2, \cdots, X_m$ are C-antichain-convex. Moreover suppose that X_1 is C-upward. Then $X_1 + X_2 + \cdots + X_m$ is convex and C-upward.

Given finitely many C-antichain-convex sets, this proposition asserts that the sum of them is convex if at least one of them is C-upward. Thus, it suffices to examine at least one of those is C-upward when all other sets are C-antichain-convex sets.

To illustrate Proposition 2. 7, let us give some examples of sets in R^2. The cone C in R^2 is depicted in Figure 1. In Figure 2, a set X_1 that satisfies both C-antichain-convexity and C-upwardness is illustrated. According to Proposition 2. 5, since $0 \in C$ and X_1 is C-upward, we have $X_1 + C = X_1$. Figure 3 depicts a set X_2 that satisfies C-antichain-convexity. Although X_2 is not convex due to the inward kink at $(-2, 3.25)$, it is still C-antichain-convex[3).

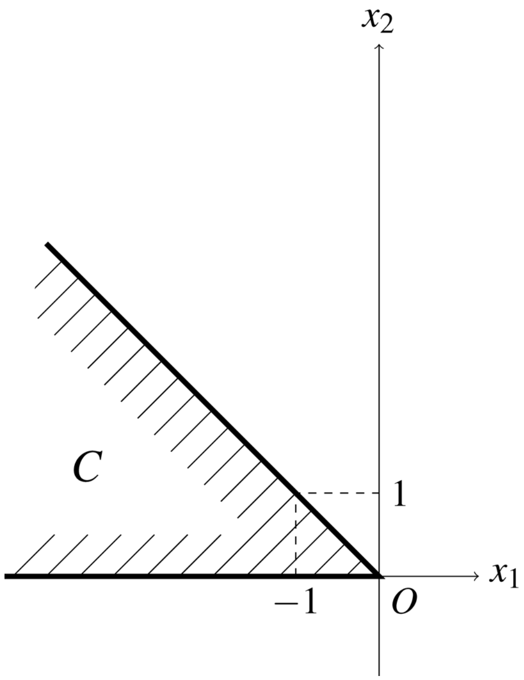

Figure 1 The set C is a cone.

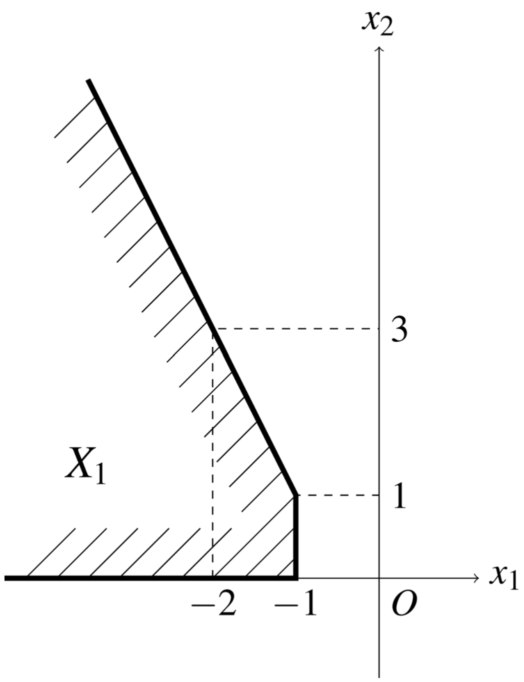

Figure 2 The set X_1 is C-antichain-convex and C-upward.

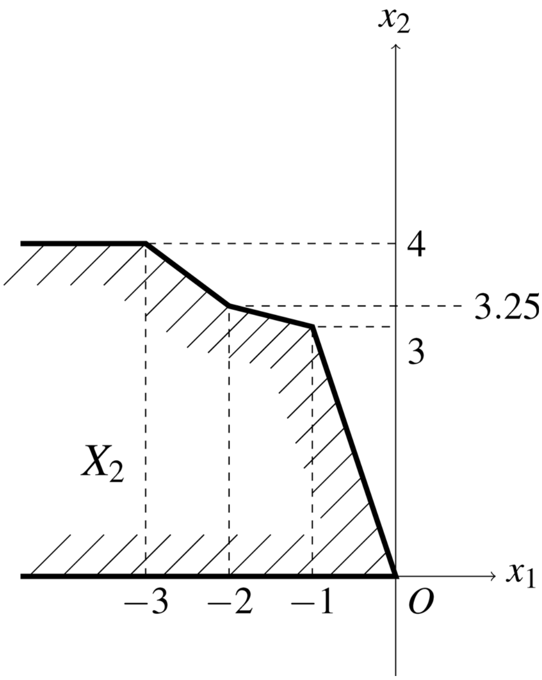

Figure 3　The set X_2 is C-antichain-convex.

Finally the sum of X_1 and X_2, illustrated in Figure 4, is indeed convex and C-upward.

3　Model and Result

This section presents the model and the main result concerning the convexity of the aggregate production set without free disposal. The individual production set is denoted by $Y_i \subset R^n$ $(i=1, 2, \cdots, n)$. The aggregate production set Y is defined as the sum of individual production sets, i.,e.,

$$Y \equiv \sum_{i=1}^{n} Y_i.$$

The set $C \subset R^n$ denotes a cone in R^n satisfying $0 \in C$.

To extend the result on the convexity of the aggregate production set established by Ceparano and Quartieri (2019) and to eliminate the assumption of free disposal as in Hamano (2022), the following modified C-antichain-convexity under star-shapedness is required.

Definition 3. 1 (the modified C-antichain-convexity under the star-shapedness)

Suppose that X is strictly star-shaped with the center $x^c \in X$; i.e., there exist a real

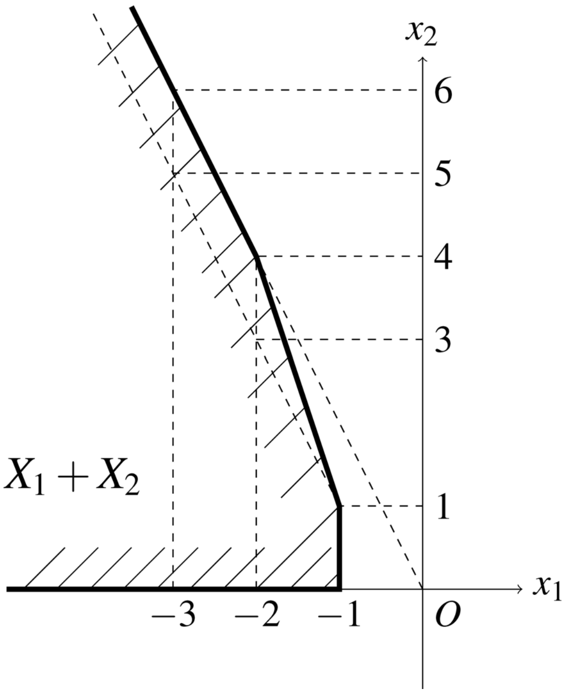

Figure 4 The sum X_1+X_2 is convex.

positive number $\varepsilon>0$ and a neighborhood $B_\varepsilon(x^c)\subset\mathrm{Int}(X)$ of x^c such that, for all $x\in B_\varepsilon(x^c)$, for all $x\in X$, and for all $t\in[0,1]$, $tx+(1-t)x^c\in X$. The set X is said to satisfy the modified C-antichain-convexity under the star-shapedness if X satisfies the following property:

> for all $x', x''\in X$ such that $x''-x'\notin C\cup(-C)$, if there is $\hat{\lambda}\in[0,1]$ such that $\hat{\lambda}x'+(1-\hat{\lambda})x''\notin X$, then there are $\bar{\lambda}\in[0,1]$, $\bar{\mu}\in[0,1]$ and $\bar{x}', \bar{x}''\in X$ with $\bar{x}''-\bar{x}'\in C\cup(-C)$, such that $\hat{\lambda}x'+(1-\hat{\lambda})x''=\bar{\mu}x^c+(1-\bar{\mu})\{\bar{\lambda}\bar{x}'+(1-\bar{\lambda})\bar{x}''\}$.

To illustrate the modified C-antichain-convexity under the star-shapedness, we provide an example of a cone C and a set X both of which do not satisfy free disposal. Figure 5 depicts the cone C in R^2. Given C, the set X is illustrated in Figure 6. It is easily verified that the set X does not satisfy the condition of the C-antichain-convexity. However, as is illustrated in Figure 7 the set X satisfies the modified C-antichain-convexity under the star-shapedness.

We now present the main result on the convexity of the aggregate production set under the modified C-antichain-convexity, which is an extension of Proposition 2. 7 (Ceparano and Quartieri (2019, Corollary 1)). We prove this in the following section.

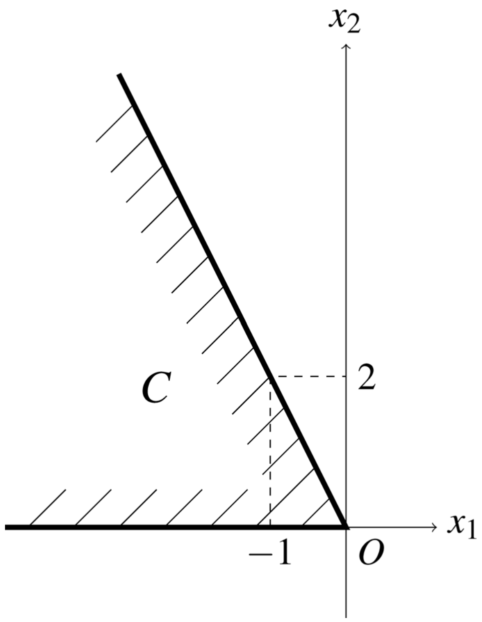

Figure 5　The set C is a cone.

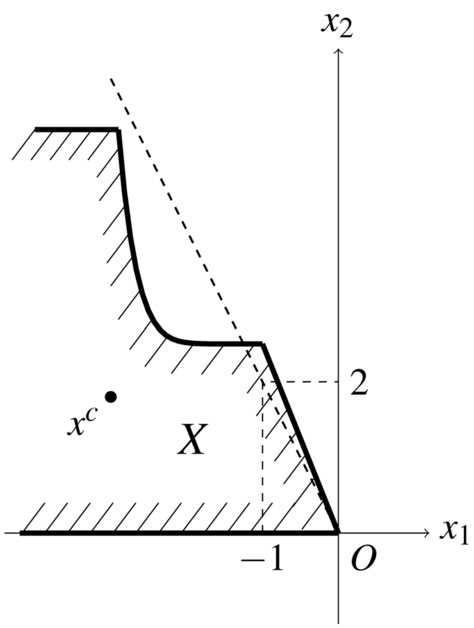

Figure 6　The set X is non-convex and does not satisfy the free disposal condition.

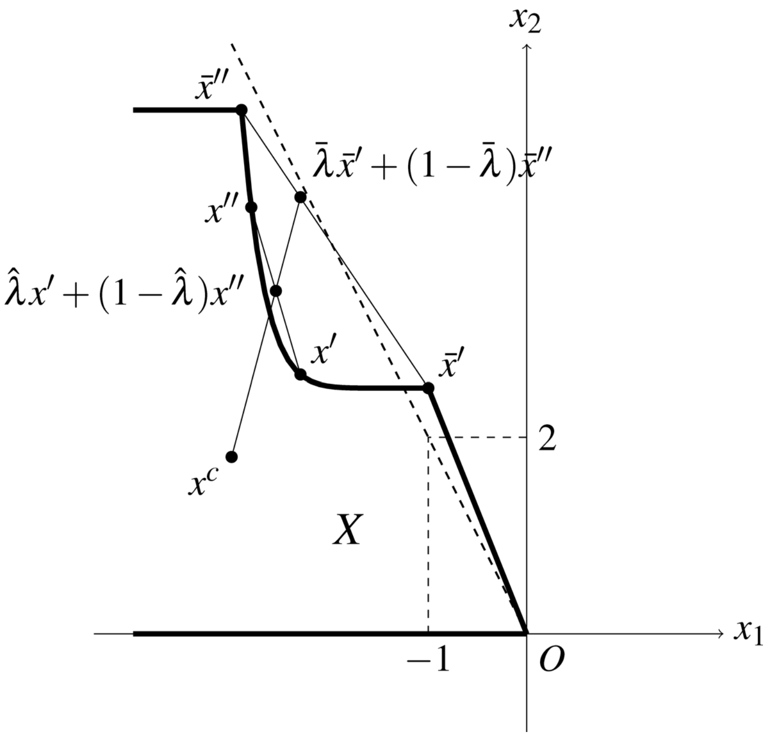

Figure 7 The set X satisfies the modified C-antichain-convexity under the star-shapedness.

Theorem 3. 2

Let C and Y_i for all $i=1, 2, \cdots, n$ be subsets in R^n. Assume that:

(i) The set C is a convex cone with $0 \in C$.

(ii) The set Y_1 is C-upward and C-antichain-convex.

(iii) For each $i=2, \cdots, n$, Y_i is strictly star-shaped with the center y_i^c and satisfies the modified C-antichain-convexity under the star-shapedness.

Then, $Y=\sum_{i=1}^{n} Y_i$ is convex and C-upward.

4 Proof of Theorem 3. 2

If the sum of sets Y_1+Y_2 is shown to be convex and C-upward, then by repeating this argument, we can conclude that $Y=\sum_{i=1}^{n} Y_i$ is also convex and C-upward.

It is easily checked that Y_1 is convex[4)]. It follows directly from Proposition 2. 6 that Y_1+Y_2 is C-upward. Additionally, according to Proposition 2. 5, C-upwardness of Y_1 is equivalent to the condition that

$$Y_1+C=Y_1, \tag{1}$$

since $0\in C$.

To show that Y_1+Y_2 is convex, let $z', z''\in Y_1+Y_2$ and $\lambda\in[0,1]$. Then, there exist $y_1', y_1''\in Y_1$ and $y_2', y_2''\in Y_2 \ni z'=y_1'+y_2'$ and $z''=y_1''+y_2''$. It is necessary to show that

$$\lambda z'+(1-\lambda)z''=\lambda y_1'+(1-\lambda)y_1''+\lambda y_2'+(1-\lambda)y_2''\in Y_1+Y_2. \tag{2}$$

Given the convexity of Y_1 we have

$$\lambda y_1'+(1-\lambda)y_1''\in Y_1. \tag{3}$$

From equations (1) to (3) it suffices to show that

$$\lambda y_2'+(1-\lambda)y_2''\in Y_2+C. \tag{4}$$

Since $R^n=C\cup(-C)\cup\overline{(C\cup(-C))}$, the remaining of the proof is divided into three cases.

case 1: $y_2''-y_2'\notin C\cup(-C)$;

case 2: $y_2''-y_2'\in C$;

case 3: $y_2''-y_2'\in(-C)$.

case 1: $y_2''-y_2'\notin C\cup(-C)$

If $\lambda y_2'+(1-\lambda)y_2''\in Y_2$, then the proof of case 1 is completed. Suppose on the contrary that $\hat\lambda y_2'+(1-\hat\lambda)y_2''\notin Y_2$ for some $\hat\lambda\in[0,1]$.

It follows from Assumption (iii) that there are $\bar\lambda\in[0,1]$, $\bar\mu\in[0,1]$ and $\bar y_2', \bar y_2''\in Y_2$ with $\bar y_2''-\bar y_2'\in C\cup(-C)$, such that

$$\hat\lambda y_2'+(1-\hat\lambda)y_2''=\bar\mu y_2^c+(1-\bar\mu)\{\bar\lambda\bar y_2'+(1-\bar\lambda)\bar y_2''\}.$$

Without loss of generality it is assumed that $\bar y_2''-\bar y_2'\in C$.

It will be shown that $\hat\lambda y_2'+(1-\hat\lambda)y_2''\in Y_2+C$. Notice that

$$\bar\mu y_2^c+(1-\bar\mu)\{\bar\lambda\bar y_2'+(1-\bar\lambda)\bar y_2''\}=\bar\mu y_2^c+(1-\bar\mu)\bar y_2'+(1-\bar\mu)(1-\bar\lambda)(\bar y_2''-\bar y_2').$$

It follows, from the hypotheses that $\bar y_2'\in Y_2$ and Y_2 is strictly star-shaped, that $\bar\mu y_2^c+(1-\bar\mu)\bar y_2'\in Y_2$. It also follows from the hypothesis that $\bar y_2''-\bar y_2'\in C$ where C is a cone, that $(1-\bar\mu)(1-\bar\lambda)(\bar y_2''-\bar y_2')\in C$. Therefore, it is concluded that

$$\lambda y_2' + (1-\lambda) y_2'' \in Y_2 + C.$$

This completes the proof of case 1.

case 2: $y_2'' - y_2' \in C$

Since C is a cone, it follows that $(1-\lambda)(y_2''-y_2') \in C$. Note also that $y_2' \in Y_2$. Therefore, it is concluded that

$$\lambda y_2' + (1-\lambda) y_2'' = y_2' + (1-\lambda)(y_2'' - y_2') \in Y_2 + C.$$

case 3: $y_2'' - y_2' \in (-C)$

Note that $y_2'' - y_2' \in (-C)$ is equivalent to $y_2' - y_2'' \in C$, which implies $(1-\lambda)(y_2' - y_2'') \in C$ because C is a cone. Then, the same argument in case 2 can be applied.

This completes the proof. **Q. E. D.**

Notes

1) See, for example, Mas-Colell et al. (1995).
2) In Ceparano and Quartieri (2019) the commodity space is a real vector space. However, in this paper it is confined to be R^n although the extension may not be difficult.
3) The slope of the line passing through any two different points between $(-1, 3)$ and $(-3, 4)$ in Figure 3 is less steep than one which is equal to the slope of the line segment through O and $(-1, 1)$ on the boundary of the set C in Figure 1.
4) Assumption (ii) and Proposition 2.4 imply that Y_1 is C-chain-convex. Consequently, by Proposition 2.3, Y_1 can be concluded to be convex.

References

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