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Existence, Comparison, and Uniqueness of the Epstein–Zin Stochastic Differential Utility with Unit Elasticity of Intertemporal Substitution*

Yuki Shigeta†

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Abstract

In this study, we examine the existence, uniqueness, and a comparison theorem for Epstein–Zin stochastic differential utility (SDU) with unit elasticity of intertemporal substitution (EIS) under mild integrability conditions. In a Brownian-jump setting with finite-activity, we establish existence on a broader class of consumption processes than previously shown in the literature. We also demonstrate that robust utility with a logarithmic felicity function subject to relative entropy costs is observationally equivalent to the unit EIS Epstein–Zin SDU. Beyond the finite-horizon analysis, we present conditions for existence in the infinite-horizon case. Finally, we consider an optimal consumption-investment problem in incomplete markets and establish a verification theorem for the associated Hamilton–Jacobi–Bellman equation.

Key words: Epstein–Zin stochastic differential utility, unit elasticity of intertemporal substitution, optimal consumption-investment problem, robust utility, jump processes

Mathematics Subject Classification (2020): 49L20, 60H20, 91B16, 91G10, 93E20

JEL Classification: C61, D81, G11

1 Introduction

The Epstein–Zin utility developed by Epstein and Zin (1989) is a class of discrete-time recursive utilities that separates time preference from risk preference. Its continuous-time counterpart, the Epstein–Zin stochastic differential utility (Epstein–Zin SDU), is proposed by Duffie and Epstein (1992). The corresponding Hamilton–Jacobi–Bellman (HJB) equation often becomes significantly simpler when the elasticity of intertemporal substitution is one (unit EIS), a case in which income and substitution effects on consumption offset each other. Owing to this tractability, many applied studies such as Chacko and Viceira (2005), Wachter (2013), Tsai and Wachter (2015), and Wachter and Zhu (2025) adopt the case of a unit EIS. Nevertheless, the mathematical foundation of the unit

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EIS Epstein–Zin SDU has received comparatively less attention since the seminal work of Schroder and Skiadas (1999). Furthermore, the assumptions in Schroder and Skiadas (1999) for the existence and uniqueness of the utility process are more demanding than the integrability conditions available for the general EIS case, as established in Kraft, Seiferling, and Seifried (2017) and Xing (2017). Similarly, the comparison theorem in Schroder and Skiadas (1999) is developed in a complete-market setting, and even there it compares the utility from the optimal consumption plan with that from an alternative consumption plan. Hence, it does not allow a comparison between utility processes associated with two arbitrary consumption processes.

Motivated by the above, this study develops the Epstein–Zin SDU for the unit-EIS case along three dimensions. First, we establish existence under jump uncertainty and mild integrability conditions, relaxing the "all-exponents" moment requirement in Schroder and Skiadas (1999). Second, we prove a comparison theorem for utility processes associated with two arbitrary consumption processes. Then, we derive uniqueness within the comparison class characterized by our comparison theorem. In this sense, our comparison-based uniqueness complements the fixed-point uniqueness result of Schroder and Skiadas (1999).

Let us briefly review the conditions assumed in Schroder and Skiadas (1999). Under a purely Brownian environment, Schroder and Skiadas (1999) assume that a progressively measurable and right-continuous consumption process C on a finite horizon $[0, T]$ satisfies the following inequality for *any* constant $l \in \mathbb{R}$:

$$\mathbb{E}\left[\int_0^T |C_t|^l \mathrm{d}t\right] < \infty.$$

In the finite-horizon case, this assumption may be acceptable because it is satisfied when consumption follows a geometric Brownian motion. However, in the infinite-horizon case ($T = \infty$), the "all-exponents" requirement is somewhat strong. Even if a discount factor is introduced, integrability for every $l \in \mathbb{R}$ may be overly demanding in standard specifications such as a geometric Brownian motion, because the growth rate of $\mathbb{E}[|C_t|^l]$ is unbounded as $|l| \to \infty$. This motivates our integrability conditions that remain compatible with the infinite-horizon construction. In this study, we impose integrability at the exponent $l = 1-\gamma$ (γ is the coefficient of relative risk aversion) together with a logarithmic growth condition.

The SDU is often characterized as a solution to backward stochastic differential equations (BSDEs). The BSDE is determined by the aggregator and the terminal condition. However, the unit EIS Epstein–Zin SDU involves a non-linear aggregator that is neither Lipschitz nor monotone in the utility index. Standard BSDE theory relies on the Lipschitzness or monotonicity of the aggregator (for example, see Delong (2013)). Therefore, it may be difficult to directly apply the existence result for BSDEs to demonstrate the existence of the SDU.

To address this issue, we employ a monotone transformation of the SDU. By applying an appropriate transformation, the utility index follows a quadratic–exponential growth BSDE (with finite-activity jumps). Indeed, Schroder and Skiadas (1999) also utilize a monotone transformation of the SDU. However, Schroder and Skiadas (1999) do not use BSDE arguments and instead rely on fixed point theory. To employ the fixed point argument, the aforementioned "all-exponents" integrability condition is required. Meanwhile, in this study we directly solve the BSDEs for the transformed utility index owing to recent advances in the theory of quadratic–exponential growth BSDEs with jumps.

We also study the relationship between the unit EIS Epstein–Zin SDU and the robust utility proposed by Hansen and Sargent (2001). As discussed in Skiadas (2003), robust utility with relative entropy costs is observationally equivalent to an SDU. Indeed, Maenhout (2004) demonstrates this observational equivalence in the case of EIS $\neq 1$. In this study, we establish observational equivalence in the case of EIS $= 1$ under jump uncertainty.

Applied studies often consider an infinitely lived agent, so the utility process is defined on an infinite horizon. Recent studies such as Herdegen, Hobson, and Jerome (2023b), Herdegen, Hobson, and Jerome (2025), and Shigeta (2025) demonstrate the existence of infinite-horizon Epstein–Zin SDUs in the case of EIS $\neq 1$. In this study, we characterize the infinite-horizon unit EIS Epstein–Zin SDU as the limit of its finite-horizon counterparts as the horizon tends to infinity.

Finally, we consider a Merton problem under jump uncertainty inspired by the setting in Wachter (2013). Applying techniques similar to those in Kraft et al. (2017), we demonstrate a verification theorem for an associated HJB equation. As noted above, the HJB equation in the unit EIS case is simpler than in general cases. However, market incompleteness makes the HJB equation non-linear, so the analysis is not trivial.

After the seminal work of Duffie and Epstein (1992), numerous studies have investigated the Epstein–Zin SDU from a mathematical perspective. For the finite-horizon case, see Duffie and Lions (1992), Schroder and Skiadas (1999), Kraft, Seifried, and Steffensen (2013), Seiferling and Seifried (2016), Kraft et al. (2017), Xing (2017), Matoussi and Xing (2018). For the infinite-horizon case, see Dang (2021), Herdegen, Hobson, and Jerome (2023a), Herdegen et al. (2023b), Herdegen et al. (2025), and Shigeta (2025). However, many of the above studies consider only the case of EIS $\neq 1$, except for Schroder and Skiadas (1999). As noted earlier, the major obstacle in the unit-EIS case is the non-linearity of the aggregator. One approach to address this non-linearity is to apply a monotonic transformation to the utility index. However, even though the BSDE for the transformed utility has an aggregator that is linear in the utility index, it contains non-linear terms with respect to the coefficients of the Brownian motion and the jump measure (quadratic in the Brownian control and exponential in the jump integrand). To handle this non-linearity, Schroder and Skiadas (1999) impose a suitable terminal condition and construct a mapping to invoke a fixed point argument under strong integrability conditions. This enables them to prove the existence and uniqueness of the corresponding utility process.

After Schroder and Skiadas (1999), the literature has made substantial progress in analyzing BSDEs that exhibit the same type of non-linearity as the transformed equation: Kobylanski (2000) and Briand and Hu (2008) for Brownian quadratic BSDEs, and Royer (2006), Becherer (2006), Morlais (2009b), Morlais (2009a), Morlais (2010), Possamaï, Kazi-Tani, and Zhou (2015), and Fujii and Takahashi (2018) for Brownian-jump quadratic–exponential BSDEs. These developments allow us to analyze the BSDE characterizing the unit EIS Epstein–Zin SDU more directly. As a benchmark, we mainly use the techniques of Morlais (2009b) and Fujii and Takahashi (2018) to demonstrate the existence of the unit EIS Epstein–Zin SDU for bounded consumption. Then, our main novelty here is to extend the analysis to unbounded consumption under a broader integrability class by exploiting the unit-EIS Epstein–Zin structure.

The remainder of this paper is as follows. Section 2 introduces the baseline setup in this study and demonstrates the existence, uniqueness, comparison theorem, and economic properties of the unit EIS Epstein–Zin SDU, including concavity, observational equivalence between robust utility and the unit EIS Epstein–Zin SDU, and the infinite-horizon utility. Section 3 considers an optimal consumption-investment problem under the unit EIS Epstein–Zin SDU in incomplete markets and presents a numerical illustration for the optimal solution. Section 4 concludes, and Appendix A provides proofs of Lemmas and Propositions omitted in the main text.

2 The Epstein–Zin SDU with Unit EIS

Initially, we introduce the mathematical setup in this study. Let T be a finite positive constant. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual conditions. Further suppose that $\mathcal{F}_0$ is trivial and $\mathcal{F} = \mathcal{F}_T$. Let $\mathbb{E}$ be the expectation operator on $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathbb{E}_\tau[\cdot] := \mathbb{E}[\cdot \mid \mathcal{F}_\tau]$ for an $\mathbb{F}$-stopping time τ on $[0,T]$. On the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, let B be a d-dimensional standard Brownian motion, and let μ be an integer-valued random measure on $([0,T]\times\mathbb{R}^{d_j}, \mathcal{B}([0,T])\otimes\mathcal{B}(\mathbb{R}^{d_j}))$. d and d_j are finite positive integers. To clarify the measurable space for jumps, we write $(J, \mathcal{J}) := (\mathbb{R}^{d_j}, \mathcal{B}(\mathbb{R}^{d_j}))$. For the jump measure μ, we adopt the setting in Becherer (2006): we denote by ν the compensator of μ. Suppose that ν can be decomposed as

$$\nu(\omega, \mathrm{d}t, \mathrm{d}x) = \lambda(\omega, t, x)\zeta(\mathrm{d}x)\mathrm{d}t,$$

where λ is a predictable, bounded, non-negative function and ζ is a probability measure on $(J, \mathcal{J})$ (without loss of generality, by absorbing the total mass into λ). Hence, there exists a positive constant K_J such that

$$\int_0^T \int_J \lambda(\omega, t, x)\zeta(\mathrm{d}x)\mathrm{d}t \le K_J T,$$

almost surely (a.s.). Thus, $\nu([0,T]\times J) < \infty$ holds a.s. so the jump measure has finite activity. For notational convenience, subscripts represent the time variable $t \in [0,T]$ in the random field. Although the random field may depend on $\omega \in \Omega$, we often omit this argument in the notation for simplicity. For a real-valued predictable random field $\psi_t(x) := \psi(\omega, t, x)$ on $[0,T]\times J$, the integral of ψ with respect to μ is defined as

$$(\psi * \mu)_t := \int_0^t \int_J \psi_s(x)\mu(\mathrm{d}s, \mathrm{d}x) = \int_{[0,t]\times J} \psi(\omega, s, x)\mu(\omega, \mathrm{d}s, \mathrm{d}x), \qquad t \in [0,T].$$

Similarly, we define $((\psi * \nu)_t)_{t\in[0,T]}$ for the compensator measure ν. By the compensation formula, $\mathbb{E}[(|\psi| * \mu)_T] = \mathbb{E}[(|\psi| * \nu)_T]$ holds whenever the integrals are well-defined. In addition, if ψ is locally integrable with respect to μ and ν, then ψ is also integrable with respect to the compensated measure $\widetilde{\mu} := \mu - \nu$, and $((\psi * \widetilde{\mu})_t)_{t\in[0,T]}$ is a discontinuous local martingale. As in Becherer (2006), we further suppose the following.

Assumption 2.1 (Weak predictable representation property) *For any square-integrable $\mathbb{F}$-martingale M, there exist a d-dimensional progressively measurable process Z and a real-valued predictable random field ψ on $[0,T]\times J$ such that*

$$M_t = M_0 + \int_0^t Z_s^\top \mathrm{d}B_s + \int_0^t \int_J \psi_s(x)\, \widetilde{\mu}(\mathrm{d}s, \mathrm{d}x), \qquad t \in [0,T],$$

and

$$\mathbb{E}\left[\int_0^T \|Z_s\|^2 \mathrm{d}s + \int_0^T \int_J |\psi_s(x)|^2 \lambda_s(x)\zeta(\mathrm{d}x)\mathrm{d}s\right] < \infty.$$

Here, the superscript $\top$ denotes the transpose of a vector or matrix, and $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^d$.

Since our construction proceeds by solving a sequence of Lipschitz BSDEs in an L^2-framework via truncation and approximation, it is sufficient to impose the above representation property for square-integrable martingales.

It is well known that Assumption 2.1 holds when $\mathbb{F}$ is the augmentation of a filtration generated by a Lévy process with characteristic triplet (c, β, ν) where $c \in \mathbb{R}$ and $\beta \in \mathbb{R}^d$ are constants and ν is a deterministic measure on $(J, \mathcal{J})$ (e.g., Theorem 5.3.6 (martingale representation theorem) in Applebaum (2009)). In general settings, however, it is not clear whether Assumption 2.1 holds. Therefore, in this paper, we impose Assumption 2.1.

2.1 Existence, comparison, and uniqueness

We define an Epstein–Zin SDU with a unit EIS. Let $\gamma \in (0,1)\cup(1,\infty)$ denote the constant coefficient of relative risk aversion, and let $\delta \in (0,\infty)$ be the constant subjective discount rate. Let $f(c,v)$ be a real-valued function such that

$$f(c,v) := \delta(1-\gamma)v\left(\log c - \frac{1}{1-\gamma}\log\big((1-\gamma)v\big)\right), \tag{2.1}$$

for $(c,v) \in (0,\infty)\times\mathbb{R}$ with $(1-\gamma)v > 0$. We now present the formal definition of the Epstein–Zin SDU with a unit EIS.

Definition 2.2 (Epstein–Zin SDU with a unit EIS) *For a $(0,\infty)$-valued, progressively measurable, and right-continuous process $C = (C_t)_{t\in[0,T]}$ and an $\mathcal{F}_T$-measurable random variable ξ with $(1-\gamma)\xi > 0$ a.s., a* unit EIS Epstein–Zin SDU *of (C,ξ), denoted by $U(C,\xi) = (U_t(C,\xi))_{t\in[0,T]}$, is a progressively measurable and càdlàg process satisfying*

$$U_t(C,\xi) = \mathbb{E}_t\left[\int_t^T f\big(C_s, U_s(C,\xi)\big)\mathrm{d}s + \xi\right], \tag{2.2}$$

with $(1-\gamma)U_t(C,\xi) > 0$ a.s. for all $t \in [0,T]$, and the right-hand side is finite for all $t \in [0,T]$ a.s.

Note that $(1-\gamma)U_t(C,\xi) > 0$ implies $U_t(C,\xi) < 0$ when $\gamma > 1$ and $U_t(C,\xi) > 0$ when $\gamma < 1$. The function f in (2.1) is indeed the generator of the unit EIS Epstein–Zin SDU. To see this, consider the generator of a general Epstein–Zin SDU with EIS $\varsigma \in (0,1)\cup(1,\infty)$ given by

$$\delta\frac{(1-\gamma)v}{(1-1/\varsigma)}\left(\frac{c^{1-1/\varsigma}}{((1-\gamma)v)^{\frac{1-1/\varsigma}{1-\gamma}}} - 1\right). \tag{2.3}$$

Then, it can be seen that

$$\lim_{\varsigma\to1}\delta\frac{(1-\gamma)v}{(1-1/\varsigma)}\left(\frac{c^{1-1/\varsigma}}{((1-\gamma)v)^{\frac{1-1/\varsigma}{1-\gamma}}} - 1\right) = \delta(1-\gamma)v\left(\log c - \frac{1}{1-\gamma}\log\big((1-\gamma)v\big)\right) = f(c,v),$$

for any $(c,v) \in (0,\infty)\times\mathbb{R}$ with $(1-\gamma)v > 0$. From this relationship, (2.1) can be regarded as the generator for the Epstein–Zin SDU with unit EIS.

Schroder and Skiadas (1999) demonstrate the existence of the unit EIS Epstein–Zin SDU. We restate their main result using the notation of this paper:

Theorem 2.3 (Theorem 1 in Schroder and Skiadas (1999)) *Let $\mathbb{F}^B = (\mathcal{F}_t^B)_{t\in[0,T]}$ be the augmentation of a filtration generated by the Brownian motion B. Suppose that a consumption process $C = (C_t)_{t\in[0,T]}$ is right-continuous and $\mathbb{F}^B$ -progressively measurable, taking values in $(0,\infty)$ and that for any $l \in \mathbb{R}$,*

$$\mathbb{E}\left[\int_0^T |C_t|^l \mathrm{d}t\right] < \infty. \tag{2.4}$$

Further suppose that $1-\gamma \le \delta$ and the terminal value is $\xi = 1/(1-\gamma)$. Then, there exists a unique unit EIS Epstein–Zin SDU of (C,ξ) with $(1-\gamma)U_t(C,\xi) > 0$ a.s. for any $t \in [0,T]$.

While the result of Schroder and Skiadas (1999) is a seminal contribution that provides a mathematical foundation for analyzing the unit EIS Epstein–Zin SDU, several features of their framework are restrictive for many modern applications. First, their analysis is conducted in a purely Brownian environment and does not include jump risk, which is often used in applied studies such as Wachter (2013). Indeed, their proof exploits the fact that a monotone transformation of the utility process is a solution to a quadratic Brownian BSDE, so it does not cover settings with jumps. Second, the integrability condition (2.4) is too restrictive. Subsequent work on the non-unit EIS case has established existence and uniqueness under weaker conditions (e.g., Xing (2017)). Third, the terminal value is fixed as a constant, whereas in many applied studies, the terminal value (i.e., the bequest utility) is stochastic. Finally, their results are stated for a finite horizon, and an infinite-horizon extension is not addressed. In the following analysis, we aim to relax these assumptions. Specifically, we extend their framework to incorporate jump processes and a stochastic bequest utility under a weaker integrability condition. The infinite-horizon case is provided in Section 2.3.

Assume that the unit-EIS Epstein–Zin SDU $U(C,\xi)$ satisfies the integrability condition $\xi + \int_0^T |f(C_s, U_s(C,\xi))|\mathrm{d}s \in L^1$. Then, the defining equation (2.2) in Definition 2.2 implies that the process

$$M_t := U_t(C,\xi) + \int_0^t f\big(C_s, U_s(C,\xi)\big)\mathrm{d}s, \qquad t \in [0,T],$$

is an $\mathbb{F}$-martingale. In particular, if M is square-integrable, then Assumption 2.1 yields the existence of a progressively measurable process Z^U and a predictable random field ψ^U on $[0,T]\times J$ such that

$$U_t(C,\xi) = \xi + \int_t^T f\big(C_s, U_s(C,\xi)\big)\mathrm{d}s - \int_t^T (Z_s^U)^\top \mathrm{d}B_s - \int_t^T \int_J \psi_s^U(x)\widetilde{\mu}(\mathrm{d}s, \mathrm{d}x), \tag{2.5}$$

for any $t \in [0,T]$. Hence, under the above integrability assumptions, the unit EIS Epstein–Zin SDU can be characterized as a solution to the BSDE with jumps (2.5). As shown in Kraft et al. (2013), the generator of the Epstein–Zin SDU in the general case (2.3) is monotone in v when $(1-\gamma)/(1-1/\varsigma) \le 1$, so the standard theory for monotone BSDEs can be applied. In contrast, the generator in the unit EIS case (2.1) is neither Lipschitz nor monotone. Therefore, it may be difficult to directly solve the BSDE (2.5).

To overcome the above difficulty, we consider a monotone transformation of $U(C,\xi)$: let

$$Y_t := \frac{1}{1-\gamma}\log\Big((1-\gamma)U_t(C,\xi)\Big), \qquad t \in [0,T],$$

which is well-defined as long as $(1-\gamma)U_t(C,\xi) > 0$ and $1 + \psi_t^U(x)/U_{t-}(C,\xi) > 0$ holds ν-a.e.

Applying the generalized Ito formula to Y yields

$$Y_t = \xi^Y + \int_t^T g(s, C_s, Y_s, Z_s, \psi_s)\mathrm{d}s - \int_t^T Z_s^\top \mathrm{d}B_s - \int_t^T \int_J \psi_s(x)\widetilde{\mu}(\mathrm{d}s, \mathrm{d}x), \tag{2.6}$$

for $t \in [0, T]$, where

$$\begin{aligned}
g(t, c, y, z, \phi) &:= \delta(\log c - y) + \frac{1-\gamma}{2}\|z\|^2 + \int_J \left(\frac{e^{(1-\gamma)\phi(x)} - 1}{1-\gamma} - \phi(x) \right) \lambda_t(x)\zeta(\mathrm{d}x), \\
Z_s &:= \frac{1}{(1-\gamma)U_{s-}(C,\xi)} Z_s^U = e^{-(1-\gamma)Y_{s-}} Z_s^U, \quad \psi_s(x) := \frac{1}{1-\gamma} \log\left(1 + \frac{\psi_s^U(x)}{U_{s-}(C,\xi)}\right), \\
\xi^Y &:= \frac{1}{1-\gamma} \log\Big((1-\gamma)\xi\Big).
\end{aligned}$$

The generator g is Lipschitz in Y, so (2.6) is simpler than (2.5) in terms of Y, although Z and ψ appear in g. Schroder and Skiadas (1999) also adopt this transformation under the Brownian environment. Accordingly, instead of tackling (2.5) directly, we work with (2.6) and then verify that the inverse transformation yields a unit-EIS Epstein–Zin SDU in the sense of Definition 2.2. Hereafter, we exploit the BSDE (2.6).

We next introduce a set of admissible pairs consisting of a consumption process and a terminal value for which we establish existence. Let $\mathcal{C}_T$ be the set of $(0,\infty)$-valued, progressively measurable, and càdlàg processes on $[0,T]$, and let L_T^1 be the set of integrable and $\mathcal{F}_T$-measurable random variables. For any constant $p \in (0,1) \cup (1,\infty)$, define

$$\mathcal{C}_{p,T} := \left\{ (C,\xi) \in \mathcal{C}_T \times L_T^1 : \mathbb{E}\left[\int_0^T |C_t|^{1-p}\mathrm{d}t + |\xi| \right] < \infty \right\}, \tag{2.7}$$

$$\mathcal{C}_{1,T} := \left\{ (C,\xi) \in \mathcal{C}_T \times L_T^1 : \mathbb{E}\left[\int_0^T |\log C_t|\mathrm{d}t + \big|\log\big((1-\gamma)\xi\big)\big| \right] < \infty, \quad (1-\gamma)\xi > 0 \text{ a.s.} \right\}. \tag{2.8}$$

Furthermore, define a subset of $\mathcal{C}_{\gamma,T} \cap \mathcal{C}_{1,T}$, denoted by $\mathcal{C}_{\gamma,T}^{\mathrm{E}}$, such that for any $(C,\xi) \in \mathcal{C}_{\gamma,T}^{\mathrm{E}}$, there exists a non-negative, right-continuous, and progressively measurable process $K^{(C,\xi)} := (K_t^{(C,\xi)})_{t\in[0,T]}$ satisfying

$$\mathbb{E}_t\left[\int_t^T \delta e^{-\delta(s-t)} \Big(\frac{C_s}{C_t}\Big)^{1-\gamma} \mathrm{d}s + e^{-\delta(T-t)} \frac{(1-\gamma)\xi}{C_t^{1-\gamma}} \right] \le K_t^{(C,\xi)}, \tag{2.9}$$

for all $t \in [0,T]$ a.s. and

$$\mathbb{E}\Big[\int_0^T (1 \vee C_t^{1-\gamma})(K_t^{(C,\xi)} + 1)\log(K_t^{(C,\xi)} + 1)\mathrm{d}t \Big] < \infty. \tag{2.10}$$

The conditions (2.9) and (2.10) are required for the unit EIS Epstein–Zin SDU under a unbounded consumption process and a terminal value having the form (2.2). If C and $(1-\gamma)\xi$ are bounded above and away from zero, then $(C,\xi) \in \mathcal{C}_{\gamma,T} \cap \mathcal{C}_{1,T}$ satisfies (2.9) and (2.10). Thus, $(C,\xi) \in \mathcal{C}_{\gamma,T}^{\mathrm{E}}$ in this bounded case. For convenience, let $\mathcal{C}_T^{\mathrm{B}}$ be a subset of $\mathcal{C}_T \times L_T^1$ such that for any $(C,\xi) \in \mathcal{C}_T^{\mathrm{B}}$, C and $(1-\gamma)\xi$ are bounded above and away from zero. By the above discussion, we have $\mathcal{C}_T^{\mathrm{B}} \subset$

$\mathcal{C}^{\mathrm{E}}_{\gamma,T}$. Furthermore, a consumption process satisfying the conditions in Schroder and Skiadas (1999) (Theorem 2.3) also belongs to $\mathcal{C}^{\mathrm{E}}_{\gamma,T}$[1].

A key ingredient in the following discussions is the "sandwich" property of the Epstein–Zin SDU. Define

$$u_p(c) := \begin{cases} \dfrac{c^{1-p}}{1-p}, & p \neq 1 \\ \log c, & p = 1, \end{cases} \qquad c \in (0,\infty), \tag{2.11}$$

where $p \in (0,\infty)$ is a constant. u_p is a constant relative risk aversion (CRRA) utility (also known as a power utility) function with a coefficient p. If the unit EIS Epstein–Zin SDU exists, it is bounded between a time-aggregated CRRA utility $u_\gamma(C_t)$ and a monotone transformation of a time-aggregated logarithmic utility $\log C_t$. This a priori estimate property allows us to establish the existence of the unit EIS Epstein–Zin SDU. As a result, the following existence result holds.

Proposition 2.4 (Existence) *For any $(C,\xi) \in \mathcal{C}^{\mathrm{E}}_{\gamma,T}$, there exists a unit EIS Epstein–Zin SDU of (C,ξ) such that*

$$\begin{aligned} 0 &< \exp\left\{ \mathbb{E}_t \left[\int_t^T \delta e^{-\delta(s-t)} \log C_s^{1-\gamma} \mathrm{d}s + e^{-\delta(T-t)} \log\left((1-\gamma)\xi\right) \right] \right\} \\ &\leq (1-\gamma) U_t(C,\xi) \leq \mathbb{E}_t \left[\int_t^T \delta e^{-\delta(s-t)} C_s^{1-\gamma} \mathrm{d}s + e^{-\delta(T-t)} (1-\gamma)\xi \right], \end{aligned} \tag{2.12}$$

holds for all $t \in [0,T]$, a.s.

The proof of Proposition 2.4 is provided in Section A.1, and we offer a brief sketch here. First, we consider the BSDE (2.6) with truncated (i.e., bounded) consumption and terminal values. By the sandwich property of the Epstein–Zin SDU, we obtain uniform upper and lower bounds for an approximating sequence of solutions to (2.6). We then show that this sequence converges to a solution of the BSDE (2.6). In addition, the resulting solution can be transformed into the representation given in Definition 2.2. Second, we address the case of unbounded consumption and terminal values. From the above argument, we obtain a sequence of unit EIS Epstein–Zin SDUs associated with truncated consumption and terminal values. Under conditions (2.9) and (2.10), we then take the limit as the truncation levels increase to infinity, establishing the solution for the unbounded case.

For uniqueness, we establish a comparison theorem for the unit EIS Epstein–Zin SDU. Under boundary assumptions, the following result holds.

Proposition 2.5 (Comparison) *Fix $C \in \mathcal{C}_T$. Let U^{sup} and U^{sub} be a progressively measurable,*

---

[1] Suppose that a consumption process C satisfies the integrability condition (2.4) for all exponent $l \in \mathbb{R}$. Set

$$K_t^{(C,\xi)} := \mathbb{E}_t \left[\int_t^T \delta e^{-\delta(s-t)} \left(\frac{C_s}{C_t}\right)^{1-\gamma} \mathrm{d}s + \frac{e^{-\delta(T-t)}}{C_t^{1-\gamma}} \right], \qquad t \in [0,T].$$

From $\log(1+x) \leq x$ for any $x \geq 0$, the Young inequality, and the Jensen inequality, we can see that C and $K^{(C,\xi)}$ with $\xi = 1/(1-\gamma)$ satisfy the inequality (2.10).

càdlàg, and uniformly integrable sub/super SDU on $[0,T]$ *such that*

$$U_t^{\sup} + \int_0^t f(C_s, U_s^{\sup})\mathrm{d}s \text{ is a local supermartingale,}$$
$$U_t^{\mathrm{sub}} + \int_0^t f(C_s, U_s^{\mathrm{sub}})\mathrm{d}s \text{ is a local submartingale,}$$

with $(1-\gamma)U_t^{\sup} > 0$ *and* $(1-\gamma)U_t^{\mathrm{sub}} > 0$ *for any* $t \in [0,T]$ *a.s. Additionally, suppose that* $U_T^{\sup} \geq U_T^{\mathrm{sub}}$ *a.s.*

Further assume one of the following.

1. *The case of* $\gamma > 1$*: there exists a positive constant* K *such that* $Ku_\gamma(C_t) \geq U_t^{\mathrm{sub}}$ *a.s. for any* $t \in [0,T]$*.*
2. *The case* $\gamma < 1$*: there exists a positive constant* K *such that* $Ku_\gamma(C_t) \leq U_t^{\sup}$ *a.s. for any* $t \in [0,T]$*.*

Then, $U_t^{\sup} \geq U_t^{\mathrm{sub}}$ *holds for any* $t \in [0,T]$ *a.s.*

The proof of Proposition 2.5 is provided in Section A.2. A key ingredient for the comparison theorem is the partial derivative of the generator $f(c,v)$ with respect to v. The homothetic property of Epstein–Zin SDUs implies that for any $a, c \in (0,\infty)$ and $v \in \mathbb{R}$ with $(1-\gamma)v > 0$, $a^{1-\gamma}f(c,v) = f(ac, a^{1-\gamma}v)$ holds. Set $a = c^{-1}$, and then $f(c,v) = f(1, v/c^{1-\gamma})c^{1-\gamma}$. Hence,

$$\frac{\partial f(c,v)}{\partial v} = \frac{\partial f(1, v/c^{1-\gamma})}{\partial v},$$

for any (c,v). Thus, $\partial f(c,v)/\partial v$ is a monotone function of the ratio $v/u_\gamma(c)$ because $f(c,v)$ is convex in v when $\gamma > 1$ and concave in v when $\gamma < 1$. Therefore, we can prove the comparison theorem by applying the integration-by-parts formula and the optional sampling theorem when the process $U_t/u_\gamma(C_t)$ is bounded away from zero.

To establish uniqueness, we define a subset of $\mathcal{C}_{\gamma,T}^{\mathrm{E}}$. Let $\mathcal{C}_{\gamma,T}^{\mathrm{U}} \subseteq \mathcal{C}_{\gamma,T}^{\mathrm{E}}$ be a consumption set such that for any $(C,\xi) \in \mathcal{C}_{\gamma,T}^{\mathrm{U}}$, there exists a constant K satisfying

$$\mathbb{E}_t\left[\int_t^T \delta e^{-\delta(s-t)} \log C_s^{1-\gamma}\mathrm{d}s + e^{-\delta(T-t)}\log\left((1-\gamma)\xi\right)\right] \geq K + \log C_t^{1-\gamma}, \tag{2.13}$$

for any $t \in [0,T]$ a.s. We can easily see that $\mathcal{C}_T^{\mathrm{B}} \subset \mathcal{C}_{\gamma,T}^{\mathrm{U}}$. The following corollary then holds.

Corollary 2.6 (Uniqueness) *For any* $(C,\xi) \in \mathcal{C}_{\gamma,T}^{\mathrm{U}}$*, the unit EIS Epstein–Zin SDU of* (C,ξ) *constructed as in Proposition 2.4, denoted by* U*, satisfies that for some positive constant* $K > 0$*,* $Ku_\gamma(C_t) \geq U_t$ *for any* $t \in [0,T]$ *a.s. if* $\gamma > 1$*, and* $Ku_\gamma(C_t) \leq U_t$ *for any* $t \in [0,T]$ *a.s. otherwise. Thus, under the consumption set* $\mathcal{C}_{\gamma,T}^{\mathrm{U}}$*, the unit EIS Epstein–Zin SDU exists and is unique in the class where* $U_t/u_\gamma(C_t)$ *is bounded away from zero.*

From inequalities (2.12) and (2.13), we obtain the ratio bound of the unit-EIS Epstein–Zin SDU in Proposition 2.5. Thus, the proof of Corollary 2.6 is immediate, so we omit it. We next discuss economic properties of the unit EIS Epstein–Zin SDU.

Proposition 2.7 *On $\mathcal{C}^{\mathrm{U}}_{\gamma,T}$, the unit EIS Epstein–Zin SDU constructed in Proposition 2.4 satisfies the following.*

1. (Homotheticity) *For $(C,\xi) \in \mathcal{C}^{\mathrm{U}}_{\gamma,T}$ and any positive constant $p > 0$, the unit EIS Epstein–Zin SDU of (C,ξ) multiplied by $p^{1-\gamma}$ coincides with the unit EIS Epstein–Zin SDU of $(pC, p^{1-\gamma}\xi)$.*
2. (Monotonicity) *For $(\widetilde{C},\widetilde{\xi}),(\widehat{C},\widehat{\xi}) \in \mathcal{C}^{\mathrm{U}}_{\gamma,T}$, suppose $\widetilde{C}_t \le \widehat{C}_t$ for any $t \in [0,T]$ and $\widetilde{\xi} \le \widehat{\xi}$ a.s. Then the unit EIS Epstein–Zin SDU of $(\widehat{C},\widehat{\xi})$ is not smaller than that of $(\widetilde{C},\widetilde{\xi})$ a.s.*
3. (Concavity) *For $(\widetilde{C},\widetilde{\xi}),(\widehat{C},\widehat{\xi}) \in \mathcal{C}^{\mathrm{U}}_{\gamma,T}$ and a constant $p \in [0,1]$, suppose that $\big(p\widetilde{C} + (1-p)\widehat{C}, u_\gamma(pu_\gamma^{-1}(\widetilde{\xi}) + (1-p)u_\gamma^{-1}(\widehat{\xi}))\big) \in \mathcal{C}^{\mathrm{U}}_{\gamma,T}$. In the case of $\gamma > 1$, further assume $(p\widetilde{C} + (1-p)\widehat{C}, p\widetilde{\xi} + (1-p)\widehat{\xi})$ satisfies (2.9) and (2.10). Then, the following inequality holds:*

$$pU_t(\widetilde{C},\widetilde{\xi}) + (1-p)U_t(\widehat{C},\widehat{\xi}) \le U_t\big(p\widetilde{C} + (1-p)\widehat{C}, u_\gamma(pu_\gamma^{-1}(\widetilde{\xi}) + (1-p)u_\gamma^{-1}(\widehat{\xi}))\big),$$

for any $t \in [0,T]$ a.s.

Section A.3 provides the proof of Proposition 2.7. Given our comparison result (Proposition 2.5), the proofs of Proposition 2.7 are straightforward. This result is an important contribution, as Duffie and Epstein (1992) only addresses the case of a Lipschitz aggregator, whereas our framework covers the unit EIS Epstein–Zin SDU, whose aggregator is not Lipschitz.

2.2 Observational equivalence between a unit EIS Epstein–Zin SDU and a robust utility with a log felicity function and relative entropy costs

We examine the observational equivalence between a unit EIS Epstein–Zin SDU and a robust utility with a log felicity function and relative entropy costs. In the robust utility framework of Hansen and Sargent (2001), the agent acknowledges possible model misspecification and evaluates utility under the worst-case distribution, while deviations from the reference model are penalized by relative entropy costs. In this sense, the robust utility represents the preferences of a pessimistic agent. Skiadas (2003) shows that, in a Brownian setting, the robust utility is observationally equivalent to a recursive stochastic differential utility. In this subsection, we extend this result by demonstrating that this equivalence also holds in the Brownian–jump setting.

We define $\mathbb{M}^\infty$ as the set of pairs (Z,ψ) where Z is a d-dimensional progressively measurable process and ψ is a real-valued predictable random field on $[0,T]\times J$ such that, for each $(Z,\psi) \in \mathbb{M}^\infty$, there exist constants $K > 0$ and $\epsilon > 0$ satisfying

$$\int_0^T \|Z_t\|^2 \mathrm{d}t + \int_0^T \int_J |\psi_t(x)|^2 \lambda_t(x)\zeta(\mathrm{d}x)\mathrm{d}t \le K \quad \text{a.s.,}$$
$$\psi_t(x) \ge -1+\epsilon, \quad \mathrm{d}\mathbb{P}\otimes\mathrm{d}\nu\text{-a.e. on } [0,T]\times J.$$

For any $(Z,\psi) \in \mathbb{M}^\infty$, consider the right-continuous local martingale M defined by

$$M_t := \int_0^t Z_s^\top \mathrm{d}B_s + \int_0^t \int_J \psi_s(x)\widetilde{\mu}(\mathrm{d}s,\mathrm{d}x), \quad t \in [0,T]. \tag{2.14}$$

M is indeed a square-integrable martingale since $(Z,\psi) \in \mathbb{M}^\infty$. Define the Doléans-Dade (stochastic) exponential of M as

$$\mathcal{E}_t(M) := \exp\left\{M_t - \frac{1}{2}\langle M^c\rangle_t\right\} \prod_{0<s\le t}\Big(1+\Delta M_s\Big)e^{-\Delta M_s}, \quad t\in[0,T], \tag{2.15}$$

where M^c denotes the continuous part of M, $\langle M^c\rangle$ denotes its quadratic variation, and $\Delta M_t = M_t - M_{t-}$. It is well known that $\mathcal{E}(M)$ solves the following SDE:

$$\mathrm{d}\mathcal{E}_t(M) = \mathcal{E}_{t-}(M)\mathrm{d}M_t, \quad \mathcal{E}_0(M) = 1.$$

In standard change-of-measure arguments, the stochastic exponential $\mathcal{E}(M)$ can be regarded as the likelihood ratio process. If $\mathcal{E}(M)$ is strictly positive and uniformly integrable, this measure change is valid owing to the Girsanov theorem. Under the new probability measure, the drift and jump compensator of a stochastic process are shifted. A central issue is whether $\mathcal{E}(M)$ is strictly positive and uniformly integrable. It is well known that the Kazamaki condition and related criteria provide sufficient conditions.

Proposition 2.8 *For any $(Z,\psi)\in\mathbb{M}^\infty$, let M be the right-continuous square-integrable martingale defined in (2.14). Then, the stochastic exponential of M, $\mathcal{E}(M)$, is strictly positive and uniformly integrable. Furthermore, there exists a probability measure $\mathbb{Q}$ on $(\Omega,\mathcal{F})$, equivalent to $\mathbb{P}$, such that $\mathcal{E}(M)$ is the Radon–Nikodym derivative process of $\mathbb{Q}$ with respect to $\mathbb{P}$:*

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \mathcal{E}_T(M).$$

On the probability space $(\Omega,\mathcal{F},\mathbb{Q})$, $B_t - \int_0^t Z_s\mathrm{d}s$ is a d-dimensional Brownian motion and the compensator of μ is $(1+\psi_t(x))\nu(\mathrm{d}t,\mathrm{d}x)$. Equivalently, $\widetilde{\mu}(\mathrm{d}t,\mathrm{d}x) - \psi_t(x)\nu(\mathrm{d}t,\mathrm{d}x)$ is a $\mathbb{Q}$-local martingale random measure.

Let $M^{(\widetilde{Z},\widetilde{\psi})}$ be the right-continuous martingale defined in (2.14) with a pair of coefficient processes $(\widetilde{Z},\widetilde{\psi})\in\mathbb{M}^\infty$. Let $\mathbb{Q}^{(\widetilde{Z},\widetilde{\psi})}$ be the probability measure on $(\Omega,\mathcal{F})$ defined by $\mathrm{d}\mathbb{Q}^{(\widetilde{Z},\widetilde{\psi})} := \mathcal{E}_T(M^{(\widetilde{Z},\widetilde{\psi})})\mathrm{d}\mathbb{P}$. Let $\mathbb{E}^{(\widetilde{Z},\widetilde{\psi})}$ be the expectation operator under $\mathbb{Q}^{(\widetilde{Z},\widetilde{\psi})}$. For any $(\widetilde{Z},\widetilde{\psi})\in\mathbb{M}^\infty$, define the relative entropy cost between $\mathbb{P}$ and $\mathbb{Q}^{(\widetilde{\mathbb{Z}},\widetilde{\psi})}$ by

$$\mathcal{R}_t(\widetilde{Z},\widetilde{\psi}) := \mathbb{E}_t^{(\widetilde{Z},\widetilde{\psi})}\left[\int_t^T \delta e^{-\delta(s-t)}\log\frac{\mathcal{E}_s(M^{(\widetilde{Z},\widetilde{\psi})})}{\mathcal{E}_t(M^{(\widetilde{Z},\widetilde{\psi})})}\mathrm{d}s + e^{-\delta(T-t)}\log\frac{\mathcal{E}_T(M^{(\widetilde{Z},\widetilde{\psi})})}{\mathcal{E}_t(M^{(\widetilde{Z},\widetilde{\psi})})}\right], \quad t\in[0,T].$$

Then, the robust utility of $C\in\mathcal{C}_T$ under the log felicity function is

$$Y_t^{\mathrm{rob.},\theta}(C) := \inf_{(\widetilde{Z},\widetilde{\psi})\in\mathbb{M}^\infty}\left\{\mathbb{E}_t^{(\widetilde{Z},\widetilde{\psi})}\left[\int_t^T \delta e^{-\delta(s-t)}\log C_s\mathrm{d}s + e^{-\delta(T-t)}\log C_T\right] + \theta\mathcal{R}_t(\widetilde{Z},\widetilde{\psi})\right\}, \tag{2.16}$$

where $\theta>0$ is a constant. The robust utility (2.16) can be written in a simpler form. From the generalized Ito formula, we have

$$
\begin{aligned}
\mathrm{d}\log\mathcal{E}_t(M^{(\widetilde{Z},\widetilde{\psi})}) &= \left[-\frac{1}{2}\|\widetilde{Z}_t\|^2 + \int_J \left(\log\left(1+\widetilde{\psi}_t(x)\right) - \widetilde{\psi}_t(x)\right)\lambda_t(x)\zeta(\mathrm{d}x)\right]\mathrm{d}t \\
&\quad + \widetilde{Z}_t^\top \mathrm{d}B_t + \int_J \log\left(1+\widetilde{\psi}_t(x)\right)\widetilde{\mu}(\mathrm{d}t,\mathrm{d}x) \\
&= \left[\frac{1}{2}\|\widetilde{Z}_t\|^2 + \int_J \left(\left(1+\widetilde{\psi}_t(x)\right)\log\left(1+\widetilde{\psi}_t(x)\right) - \widetilde{\psi}_t(x)\right)\lambda_t(x)\zeta(\mathrm{d}x)\right]\mathrm{d}t \\
&\quad + \widetilde{Z}_t^\top \mathrm{d}B_t^{(\widetilde{Z},\widetilde{\psi})} + \int_J \log\left(1+\widetilde{\psi}_t(x)\right)\widetilde{\mu}^{(\widetilde{Z},\widetilde{\psi})}(\mathrm{d}t,\mathrm{d}x),
\end{aligned}
$$

where $(B^{(\widetilde{Z},\widetilde{\psi})}, \widetilde{\mu}^{(\widetilde{Z},\widetilde{\psi})})$ denotes the Brownian motion and compensated random measure under $\mathbb{Q}^{(\widetilde{Z},\widetilde{\psi})}$. Since $(\widetilde{Z},\widetilde{\psi}) \in \mathbb{M}^\infty$, the stochastic integrals are true martingales under $\mathbb{Q}^{(\widetilde{Z},\widetilde{\psi})}$. Let

$$
R_t(\widetilde{Z}_t,\widetilde{\psi}_t) := \frac{1}{2}\|\widetilde{Z}_t\|^2 + \int_J \left(\left(1+\widetilde{\psi}_t(x)\right)\log\left(1+\widetilde{\psi}_t(x)\right) - \widetilde{\psi}_t(x)\right)\lambda_t(x)\zeta(\mathrm{d}x), \quad t \in [0,T].
$$

Then,

$$
\begin{aligned}
\mathcal{R}_t(\widetilde{Z},\widetilde{\psi}) &= \mathbb{E}_t^{(\widetilde{Z},\widetilde{\psi})}\left[\int_t^T \delta e^{-\delta(s-t)}\log\frac{\mathcal{E}_s(M^{(\widetilde{Z},\widetilde{\psi})})}{\mathcal{E}_t(M^{(\widetilde{Z},\widetilde{\psi})})}\mathrm{d}s + e^{-\delta(T-t)}\log\frac{\mathcal{E}_T(M^{(\widetilde{Z},\widetilde{\psi})})}{\mathcal{E}_t(M^{(\widetilde{Z},\widetilde{\psi})})}\right] \\
&= \mathbb{E}_t^{(\widetilde{Z},\widetilde{\psi})}\left[\int_t^T \delta e^{-\delta(s-t)}\int_t^s R_r(\widetilde{Z}_r,\widetilde{\psi}_r)\mathrm{d}r\mathrm{d}s + e^{-\delta(T-t)}\int_t^T R_r(\widetilde{Z}_r,\widetilde{\psi}_r)\mathrm{d}r\right] \\
&= \mathbb{E}_t^{(\widetilde{Z},\widetilde{\psi})}\left[\int_t^T \left(\int_r^T \delta e^{-\delta(s-t)}\mathrm{d}s\right) R_r(\widetilde{Z}_r,\widetilde{\psi}_r)\mathrm{d}r + e^{-\delta(T-t)}\int_t^T R_r(\widetilde{Z}_r,\widetilde{\psi}_r)\mathrm{d}r\right] \\
&= \mathbb{E}_t^{(\widetilde{Z},\widetilde{\psi})}\left[\int_t^T e^{-\delta(s-t)} R_s(\widetilde{Z}_s,\widetilde{\psi}_s)\mathrm{d}s\right], \quad t \in [0,T].
\end{aligned}
$$

Thus, (2.16) can be expressed as

$$
Y_t^{\mathrm{rob.},\theta}(C) = \inf_{(\widetilde{Z},\widetilde{\psi})\in\mathbb{M}^\infty} \mathbb{E}_t^{(\widetilde{Z},\widetilde{\psi})}\left[\int_t^T e^{-\delta(s-t)}\left(\delta\log C_s + \theta R_s(\widetilde{Z}_s,\widetilde{\psi}_s)\right)\mathrm{d}s + e^{-\delta(T-t)}\log C_T\right]. \quad (2.17)
$$

Then, the following observational equivalence result holds.

Proposition 2.9 *Let $\theta > 0$ be a constant. For any $(C, u_{1+1/\theta}(C_T)) \in \mathcal{C}_T^{\mathrm{B}}$, let $U^{\mathrm{EZ},1+1/\theta}(C)$ be an unit EIS Epstein–Zin SDU of $(C, u_{1+1/\theta}(C_T))$ with RRA $1+1/\theta$. Then, $u_{1+1/\theta}(\exp\{Y_t^{\mathrm{rob.},\theta}(C)\}) = U_t^{\mathrm{EZ},1+1/\theta}(C)$ a.s. on $(\Omega,\mathcal{F},\mathbb{P})$ for any $t \in [0,T]$.*

The proof of Proposition 2.9 is provided in Section A.4. The robust utility represents the objective function of a pessimistic agent. In contrast, we may define an objective function for an optimistic agent as

$$
Y_t^{\mathrm{opt.},\theta}(C) := \sup_{(\widetilde{Z},\widetilde{\psi})\in\mathbb{M}^\infty}\left\{\mathbb{E}_t^{(\widetilde{Z},\widetilde{\psi})}\left[\int_t^T \delta e^{-\delta(s-t)}\log C_s\mathrm{d}s + e^{-\delta(T-t)}\log C_T\right] - \theta\mathcal{R}_t(\widetilde{Z},\widetilde{\psi})\right\}, \quad (2.18)
$$

where $\theta > 0$ is a constant. From an argument analogous to the proof of Proposition 2.9, we obtain the following corollary.

Corollary 2.10 *Let $\theta > 1$ be a constant. For any $(C, u_{1-1/\theta}(C_T)) \in \mathcal{C}_T^{\mathrm{B}}$, let $U^{\mathrm{EZ},1-1/\theta}(C)$ denote the unit EIS Epstein–Zin SDU of $(C, u_{1-1/\theta}(C_T))$ with RRA $1-1/\theta$. Then, $u_{1-1/\theta}(\exp\{Y_t^{\mathrm{opt.},\theta}(C)\}) = U_t^{\mathrm{EZ},1-1/\theta}(C)$ a.s. on $(\Omega, \mathcal{F}, \mathbb{P})$ for any $t \in [0, T]$.*

The proof of Corollary 2.10 is essentially the same as that of Proposition 2.9, so it is omitted.

2.3 Infinite horizon

We now turn to the infinite-horizon unit-EIS Epstein–Zin SDU. We begin with a formal definition.

Definition 2.11 (Infinite-horizon unit EIS Epstein–Zin SDU) *For any consumption process $C = (C_t)_{t\in[0,\infty)}$, a progressively measurable and càdlàg process $U(C) = (U_t(C))_{t\in[0,\infty)}$ is an* infinite-horizon unit EIS Epstein–Zin SDU *of C if: (1) $(1-\gamma)U_t(C) > 0$ for any $t \in [0, \infty)$ a.s.; (2) it is of class DL; and (3) for any $0 \le t \le T < \infty$, it satisfies*

$$U_t(C) = \mathbb{E}_t\biggl[U_T(C) + \int_t^T f\bigl(C_s, U_s(C)\bigr)\mathrm{d}s\biggr]. \tag{2.19}$$

Next, we consider an admissible set of consumption processes. Let $\mathcal{C}_{\gamma,\infty}^{\mathrm{E}}$ denote the set of consumption processes on $[0, \infty)$ such that each $C \in \mathcal{C}_{\gamma,\infty}^{\mathrm{E}}$ is $(0, \infty)$- valued, right-continuous, progressively measurable, and satisfies the following two integrability conditions: (1) the infinite-horizon integrability condition:

$$\mathbb{E}\biggl[\int_0^\infty e^{-\delta t}\Bigl(C_t^{1-\gamma} + |\log C_t|\Bigr)\mathrm{d}t\biggr] < \infty.$$

(2) the finite-horizon integrability condition: there exists a non-negative, right-continuous, and progressively measurable process $K^C = (K_t^C)_{t\ge0}$ on $[0, \infty)$ (depending on C) such that, for any finite $t \in [0, \infty)$, it satisfies

$$\mathbb{E}_t\biggl[\int_t^\infty \delta e^{-\delta(s-t)}\biggl(\frac{C_s}{C_t}\biggr)^{1-\gamma}\mathrm{d}s\biggr] \le K_t^C,$$

and

$$\mathbb{E}\biggl[\int_0^t \bigl(1 \vee C_s^{1-\gamma}\bigr)(K_s^C + 1)\log(K_s^C + 1)\mathrm{d}s\biggr] < \infty.$$

The following proposition establishes the existence of the infinite-horizon unit EIS Epstein–Zin SDU on $\mathcal{C}_{\gamma,\infty}^{\mathrm{E}}$.

Proposition 2.12 *For any $C \in \mathcal{C}_{\gamma,\infty}^{\mathrm{E}}$, there exists an infinite-horizon unit EIS Epstein–Zin SDU of C, denoted by $U(C)$. It satisfies*

$$0 < \exp\biggl\{\mathbb{E}_t\biggl[\int_t^\infty \delta e^{-\delta(s-t)}\log C_s^{1-\gamma}\mathrm{d}s\biggr]\biggr\} \le (1-\gamma)U_t(C) \le \mathbb{E}_t\biggl[\int_t^\infty \delta e^{-\delta(s-t)}C_s^{1-\gamma}\mathrm{d}s\biggr] < \infty, \tag{2.20}$$

for any $t \in [0, \infty)$.

The proof of Proposition 2.12 is given in Section A.5. As shown in the proof, the infinite-horizon utility in Proposition 2.12 is obtained as the limit of the finite-horizon utilities as the maturity tends to go to infinity. Consequently, the infinite-horizon utility inherits the economic properties established in the finite-horizon setting (see Proposition 2.7).

To establish uniqueness, we provide the following comparison result.

Proposition 2.13 *Suppose that there exist two infinite-horizon unit EIS Epstein–Zin SDUs of a consumption process $C \in \mathcal{C}^{\mathrm{E}}_{\gamma,\infty}$, denoted by $U^1(C)$ and $U^2(C)$, and two positive constants $\underline{K}$ and $\overline{K}$ with $0 < \underline{K} \leq \overline{K} < \infty$ such that*

$$\underline{K} \leq \frac{U^i_t(C)}{u_\gamma(C_t)} \leq \overline{K}, \tag{2.21}$$

for any $t \in [0,\infty)$ and $i = 1,2$. Then, $U^1_t(C) = U^2_t(C)$ holds for any $t \in [0,\infty)$ a.s.

Section A.6 provides the proof of Proposition 2.13.

Remark 2.14 *Combining inequality (2.20) and Proposition 2.13, the infinite-horizon Epstein–Zin SDU of $C \in \mathcal{C}^{\mathrm{E}}_{\gamma,\infty}$ is unique within the comparison class of Proposition 2.13 if there exist constants $K_1 \in \mathbb{R}$ and $K_2 \geq 0$ such that for any $t \in [0,\infty)$,*

$$\mathbb{E}_t\left[\int_t^\infty \delta e^{-\delta(s-t)} \log C_s^{1-\gamma} \mathrm{d}s\right] \geq K_1 + \log C_t^{1-\gamma}, \quad \text{and} \quad \mathbb{E}_t\left[\int_t^\infty \delta e^{-\delta(s-t)} C_s^{1-\gamma} \mathrm{d}s\right] \leq K_2 C_t^{1-\gamma}.$$

These inequalities are convenient sufficient conditions for the ratio bounds required in Proposition 2.13. Moreover, they also imply condition (2) in the definition of $\mathcal{C}^{E}_{\gamma,\infty}$. In particular, they hold when C is uniformly bounded above and away from zero.

In this section, we have examined the existence and uniqueness of the infinite-horizon unit EIS Epstein–Zin SDU, motivated by its popularity in applied research. In the next section, however, we return to the finite-horizon formulation for tractability. Because the finite-horizon unit EIS Epstein–Zin SDU converges to its infinite-horizon counterpart as the horizon goes to infinity, solving the finite-horizon problem provides a key step toward understanding the infinite-horizon case.

Remark 2.15 *In applied studies, the infinite-horizon recursive utility is often written as*

$$U_t(C) = \mathbb{E}_t\left[\int_t^\infty f\big(C_s, U_s(C)\big)\mathrm{d}s\right], \qquad t \in [0,\infty). \tag{2.22}$$

The representation (2.22) is convenient, but a fully rigorous justification typically requires additional conditions (e.g., a transversality condition ensuring that $E_t[U_T(C)] \to 0$ as $T \to \infty$). In contrast, the finite-horizon-consistent formulation (2.19) does not impose such additional conditions. Moreover, the infinite-horizon utility process can be uniquely defined even without transversality, as seen in Propositions 2.12 and 2.13. In many applications, arguments written in terms of (2.22) can be carried out under the more general formulation (2.19) with little or no change. Accordingly, we adopt (2.19) as our baseline representation for the infinite-horizon case. For related discussions, see Herdegen et al. (2023a) and Shigeta (2025).

3 An Optimal Consumption–Investment Problem

3.1 Setting and verification

We consider a finite-horizon optimal consumption-investment problem as an application. Suppose $d = 2$ and $d_j = 1$ (i.e., $J = \mathbb{R}$). Let (B^1, B^2) be a two-dimensional Brownian motion. Let η be a stochastic process satisfying

$$\mathrm{d}\eta_t = \alpha(\eta_t)\mathrm{d}t + \beta(\eta_t)\mathrm{d}B_t^1, \tag{3.1}$$

where $\alpha : \mathbb{R} \to \mathbb{R}$ and $\beta : \mathbb{R} \to \mathbb{R}$ are measurable functions. Let μ be a random measure on $([0,T] \times \mathbb{R}, \mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}))$, whose compensator ν satisfies

$$\nu(\mathrm{d}t, \mathrm{d}x) = \lambda(\eta_t)\zeta(\mathrm{d}x)\mathrm{d}t,$$

where $\lambda : \mathbb{R} \to [0, \infty)$ is a non-negative measurable function and ζ is a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Let S^0 and S denote the prices of a risk-free asset and risky asset, respectively. (S^0, S) satisfies the following SDE:

$$\begin{aligned} \mathrm{d}S_t^0 &= r(\eta_t)S_t^0\mathrm{d}t, \\ \mathrm{d}S_t^1 &= S_{t-}^1\left(m(\eta_t)\mathrm{d}t + \sigma(\eta_t)\mathrm{d}B_t^2 + \int_{\mathbb{R}} x\mu(\mathrm{d}t, \mathrm{d}x)\right), \end{aligned}$$

where r, m, and σ are measurable functions from $\mathbb{R}$ to $\mathbb{R}$. We impose assumptions on $(\alpha, \beta, \lambda, \zeta, r, m, \sigma)$ as follows.

Assumption 3.1

1. *α, β, λ, r, m, and σ are bounded, Lipschitz continuous, and continuously differentiable with Lipschitz continuous derivatives.*
2. *(β, σ) satisfies the uniform ellipticity condition:* $\inf_{\eta \in \mathbb{R}} |\beta(\eta)| > 0$ *and* $\inf_{\eta \in \mathbb{R}} |\sigma(\eta)| > 0$.
3. *$\int_{\mathbb{R}} |x|^2\zeta(\mathrm{d}x) < \infty$ and there exists a constant $\epsilon \in (0,1)$ such that $\zeta([-1+\epsilon, \infty)) = 1$.*
4. *There exists a finite constant $\underline{K}_\Pi \leq 0$ such that, for any $\eta \in \mathbb{R}$, the following inequality holds:*

$$m(\eta) - r(\eta) - \gamma\sigma^2(\eta)\underline{K}_\Pi + \lambda(\eta)\int_{\mathbb{R}} (1 + \underline{K}_\Pi x)^{-\gamma} x\zeta(\mathrm{d}x) > 0. \tag{3.2}$$

   Further suppose $1 + \underline{K}_\Pi x > 0$ for any $x \in A$ where A is a set in $\mathcal{B}(\mathbb{R})$ satisfying $\zeta(A) = 1$.
5. *There exists a finite constant $\overline{K}_\Pi \in (0, 1/(1-\epsilon))$ such that for any $\eta \in \mathbb{R}$, the following inequality holds:*

$$m(\eta) - r(\eta) - \gamma\sigma^2(\eta)\overline{K}_\Pi + \lambda(\eta)\int_{\mathbb{R}} (1 + \overline{K}_\Pi x)^{-\gamma} x\zeta(\mathrm{d}x) < 0. \tag{3.3}$$

   Further suppose that $1 + \overline{K}_\Pi x > 0$ for any $x \in A$ where A is a set in $\mathcal{B}(\mathbb{R})$ satisfying $\zeta(A) = 1$.
6. *For $p = 0, 1, 2$, the following inequality holds.*

$$\sup_{\Pi \in [\underline{K}_\Pi, \overline{K}_\Pi]} \int_{\mathbb{R}} (1 + \Pi x)^{1-\gamma-p}|x|^p\zeta(\mathrm{d}x) < \infty. \tag{3.4}$$

Furthermore, the following inequality also holds.

$$\sup_{\Pi\in[\underline{K}_\Pi,\overline{K}_\Pi]}\int_{\mathbb{R}}(1+\Pi x)^{4(1-\gamma)}\zeta(\mathrm{d}x)<\infty. \tag{3.5}$$

The conditions (3.2), (3.3), (3.4), and (3.5) are technical requirements used to ensure the existence of an admissible optimal portfolio process.

Let $C:=(C_t)_{t\in[0,T]}$ and $\Pi:=(\Pi_t)_{t\in[0,T]}$ denote a consumption process and a portfolio process, respectively. The corresponding wealth process $W:=(W_t)_{t\in[0,T]}$ controlled by (C,Π) satisfies

$$\begin{aligned}\mathrm{d}W_t &= W_{t-}\Big((1-\Pi_t)\frac{\mathrm{d}S_t^0}{S_t^0}+\Pi_{t-}\frac{\mathrm{d}S_t^1}{S_{t-}^1}\Big)-C_t\mathrm{d}t\\ &=\Big[W_t\Big(r(\eta_t)+m_e(\eta_t)\Pi_t\Big)-C_t\Big]\mathrm{d}t+W_t\Pi_t\sigma(\eta_t)\mathrm{d}B_t^2+\int_{\mathbb{R}}W_{t-}\Pi_{t-}x\mu(\mathrm{d}t,\mathrm{d}x),\end{aligned}$$

where $m_e(\eta):=m(\eta)-r(\eta)$. Let $(W^{(t,w,\eta);(C,\Pi)},\eta^\eta)$ denote the wealth and state variable processes starting from $(W_t^{(t,w,\eta);(C,\Pi)},\eta_t^\eta)=(w,\eta)\in(0,\infty)\times\mathbb{R}$ and controlled by (C,Π). We define an admissible set of (C,Π) as follows.

Definition 3.2 (Admissible set) *Fix a constant $\overline{k}>0$. For any $(t,w,\eta)\in[0,T]\times(0,\infty)\times\mathbb{R}$, a pair of a consumption process and portfolio process (C,Π) is* admissible *under the initial condition (t,w,η) if it satisfies the following.*

1. *$W^{(t,w,\eta);(C,\Pi)}$ has a strong solution on $[t,T]$, taking values in $(0,\infty)$.*
2. *$(C,\overline{k}u_\gamma(W_T^{(t,w,\eta);(C,\Pi)}))\in\mathcal{C}^{\mathrm{U}}_{\gamma,T}$. Moreover, Π is bounded, right-continuous, and predictable, and satisfies a.s.*
$$\underline{K}_\Pi\le\inf_{s\in[t,T]}\Pi_s,\qquad \text{and}\qquad \sup_{s\in[t,T]}\Pi_s<\frac{1}{1-\epsilon}.$$
3. *$W^{(t,w,\eta);(C,\Pi)}$ satisfies*
$$\mathbb{E}\Big[\sup_{s\in[t,T]}\Big|W_s^{(t,w,\eta);(C,\Pi)}\Big|^{2(1-\gamma)}\Big]<\infty. \tag{3.6}$$
4. *If $\gamma<1$, then there exists a constant $l\ge1$, which may depend on (C,Π), such that $W_t^{(t,w,\eta);(C,\Pi)}\ge lC_t$ for any $t\in[0,T]$ a.s.*

We denote by $\mathcal{A}(t,w,\eta)$ the set of all admissible (C,Π) under the initial condition (t,w,η).

For any $(t,w,\eta)\in[0,T]\times(0,\infty)\times\mathbb{R}$, we define the value function as

$$V(t,w,\eta):=\sup_{(C,\Pi)\in\mathcal{A}(t,w,\eta)}U_t\big(C,\overline{k}u_\gamma(W_T^{(t,w,\eta);(C,\Pi)})\big). \tag{3.7}$$

The corresponding HJB equation for (3.7) is

$$-\frac{\partial v(t,w,\eta)}{\partial t}-\sup_{(\Pi,c)\in\mathbb{R}\times(0,\infty)}\Big\{\mathcal{L}^{\Pi,c}v(t,w,\eta)+f\big(c,v(t,w,\eta)\big)\Big\}=0, \tag{3.8}$$

with terminal condition $v(T,w,\eta) = \overline{k}u_\gamma(w)$, where

$$\begin{aligned}\mathcal{L}^{\Pi,c}v(t,w,\eta) &= \Big(w\big(r(\eta)+m_e(\eta)\Pi\big)-c\Big)\frac{\partial v(t,w,\eta)}{\partial w}+\frac{1}{2}w^2\sigma^2(\eta)\Pi^2\frac{\partial^2 v(t,w,\eta)}{\partial w^2}\\ &\quad+\alpha(\eta)\frac{\partial v(t,w,\eta)}{\partial \eta}+\frac{1}{2}\beta^2(\eta)\frac{\partial^2 v(t,w,\eta)}{\partial \eta^2}\\ &\quad+\lambda(\eta)\int_{\mathbb{R}}\Big(v\big(t,w(1+\Pi x),\eta\big)-v(t,w,\eta)\Big)\zeta(\mathrm{d}x).\end{aligned}$$

Let us consider the ansatz for the solution to the HJB equation (3.8) before verification. Suppose that the solution v has the following form.

$$v(t,w,\eta) = h(t,\eta)u_\gamma(w), \qquad (t,w,\eta)\in[0,T]\times(0,\infty)\times\mathbb{R}, \tag{3.9}$$

where h is a $C^{1,2}([0,T)\times\mathbb{R})\cap C([0,T]\times\mathbb{R})$-function. Substituting (3.9) into (3.8) yields

$$\frac{\partial h(t,\eta)}{\partial t}+\widetilde{r}(\eta)h(t,\eta)+\alpha(\eta)\frac{\partial h(t,\eta)}{\partial \eta}+\frac{1}{2}\beta^2(\eta)\frac{\partial^2 h(t,\eta)}{\partial \eta^2}-\delta h(t,\eta)\log\big(h(t,\eta)\big)=0, \tag{3.10}$$

with terminal condition $h(T,\eta)=\overline{k}$, where

$$\begin{aligned}\widetilde{r}(\eta) &:= (1-\gamma)\bigg(r(\eta)+m_e(\eta)\Pi^*(\eta)-\frac{\gamma}{2}\sigma^2(\eta)\big(\Pi^*(\eta)\big)^2+\delta(\log\delta-1)\bigg)\\ &\quad+\lambda(\eta)\int_{\mathbb{R}}\Big(\big(1+\Pi^*(\eta)x\big)^{1-\gamma}-1\Big)\zeta(\mathrm{d}x),\end{aligned}$$

and $\Pi^*(\eta)$ is the unique solution to the following nonlinear equation with respect to Π.

$$m_e(\eta)-\gamma\sigma^2(\eta)\Pi+\lambda(\eta)\int_{\mathbb{R}}(1+\Pi x)^{-\gamma}x\zeta(\mathrm{d}x)=0. \tag{3.11}$$

The uniqueness of $\Pi^*(\eta)$ follows from Assumption 3.1. Indeed, Π^* is a candidate for an optimal portfolio. The equation (3.11) is the first-order condition for optimality in the maximization problem in the HJB equation (3.8) under the ansatz (3.9). Meanwhile, $C^*(w) := \delta w$ solves the maximization problem in the HJB equation (3.8) under the ansatz (3.9). Thus, (C^*,Π^*) is a candidate for an optimal feedback control for the problem (3.7).

Remark 3.3 *If $|\Pi x|$ is sufficiently small for relevant $x\in\operatorname{supp}(\zeta)$, then the first-order approximation gives*

$$(1+\Pi x)^{-\gamma} = 1-\gamma\Pi x+o(|\Pi x|).$$

Substituting the above into the first-order condition (3.11) and collecting terms yields

$$\Pi^*(\eta)\approx\frac{m_e(\eta)+\lambda(\eta)M_1}{\gamma(\sigma^2(\eta)+\lambda(\eta)M_2)}, \qquad M_i := \int_{\mathbb{R}}x^i\zeta(\mathrm{d}x), \quad i=1,2.$$

In particular, $\Pi^*(\eta)$ approximately scales as $1/\gamma$, with the jump term shifting the effective drift by $\lambda(\eta)M_1$ and inflating the effective variance by $\lambda(\eta)M_2$.

We now examine the regularity of $\widetilde{r}$ and Π^*.

Lemma 3.4 *$\widetilde{r}$ and Π^* are Lipschitz continuous and continuously differentiable.*

The proof of Lemma 3.4 is delegated to Section A.7. In practice, it is more convenient to reformulate the reduced HJB equation (3.10) into an iterative scheme that repeatedly solves the following PDE.

$$\frac{\partial h^n(t,\eta)}{\partial t} + \widetilde{r}_{\text{stab.}}(\eta) h^n(t,\eta) + \alpha(\eta)\frac{\partial h^n(t,\eta)}{\partial \eta} + \frac{1}{2}\beta^2(\eta)\frac{\partial^2 h^n(t,\eta)}{\partial \eta^2} + f_*\big(h^{n-1}(t,\eta)\big) = 0, \tag{3.12}$$

with $h^n(T,\eta) = \overline{k}$ for any $n \in \mathbb{N}$, where

$$\widetilde{r}_{\text{stab.}}(\eta) := \widetilde{r}(\eta) - K_{\text{stab.}}, \qquad f_*(h) := -\rho_*(h)\big(\delta \log \rho_*(h) - K_{\text{stab.}}\big),$$

and $\rho_* : \mathbb{R} \to (0,\infty)$ is a bounded smooth truncation function that coincides with the identity on a large interval. It is used to keep the iteration uniformly bounded; details are given in Appendix A.8. $K_{\text{stab.}}$ is an arbitrary non-negative constant introduced as a numerical stabilization parameter. Although Lemma 3.5 also holds with $K_{\text{stab.}} = 0$, in practice we typically choose $K_{\text{stab.}}$ sufficiently large so that $\widetilde{r}_{\text{stab.}}(\eta) = \widetilde{r}(\eta) - K_{\text{stab.}} < 0$ on the computational domain, which improves numerical stability. We set $h^0(t,\eta) = \overline{k}$ for $(t,\eta) \in [0,T] \times \mathbb{R}$. Interpreting (3.12) as a PDE with respect to h^n, we see that it is a linear inhomogeneous equation, which is easier to solve numerically. We are interested in whether the sequence $\{h^n\}_{n\in\mathbb{N}}$ converges and, if so, whether its limit solves the HJB equation (3.10). The next lemma shows that it does.

Lemma 3.5 *The reduced HJB equation (3.10) has a unique classical solution in $C_b^{1,2}([0,T]\times\mathbb{R})$, denoted by h. Moreover, the sequence of solutions $\{h^n\}_{n\in\mathbb{N}}$ to the iterative PDEs (3.12) converges uniformly to h. Furthermore, there exist two constants $0 < \underline{h} \le \overline{h} < \infty$, depending only on δ, $\overline{k}$, T, and $\widetilde{r}$, such that for any $(t,\eta) \in [0,T]\times\mathbb{R}$:*

$$\underline{h} \le h(t,\eta) \le \overline{h}. \tag{3.13}$$

The proof of Lemma 3.5 is provided in Section A.8. From Lemma 3.5, the HJB equation (3.8) admits a solution $v(t,w,\eta)$ expressed through the ansatz (3.9). We now present the verification theorem for the HJB equation (3.8).

Proposition 3.6 *For any $(t_0,w,\eta) \in [0,T]\times(0,\infty)\times\mathbb{R}$, $v(t_0,w,\eta) \ge V(t_0,w,\eta)$ holds. Furthermore, $(C^*(w'),\Pi^*(\eta')) = (\delta w',\Pi^*(\eta'))$ for $(w',\eta') \in (0,\infty)\times\mathbb{R}$ is an admissible and optimal feedback control in the following sense: consider the following SDE on $[t_0,T]$:*

$$\begin{aligned} \mathrm{d}W_t^{*,(t_0,w,\eta)} &= W_t^{*,(t_0,w,\eta)}\Big(r(\eta_t^{(t_0,\eta)}) + m_e(\eta_t^{(t_0,\eta)})\Pi^*(\eta_t^{(t_0,\eta)}) - \delta\Big)\mathrm{d}t \\ &\quad + W_t^{*,(t_0,w,\eta)}\Pi^*(\eta_t^{(t_0,\eta)})\sigma(\eta_t^{(t_0,\eta)})\mathrm{d}B_t^2 + \int_{\mathbb{R}} W_{t-}^{*,(t_0,w,\eta)}\Pi^*(\eta_t^{(t_0,\eta)})x\mu(\mathrm{d}t,\mathrm{d}x), \end{aligned} \tag{3.14}$$

with $W_{t_0}^{*,(t_0,w,\eta)} = w$. Then, the SDE (3.14) has a strong solution taking values in $(0,\infty)$, and $(C^*(W^{*,(t_0,w,\eta)}),\Pi^*(\eta^{(t_0,\eta)})) \in \mathcal{A}(t_0,w,\eta)$. Moreover, it holds that

$$v(t_0,w,\eta) = V(t_0,w,\eta) = U_{t_0}\big(C^*(W^{*,(t_0,w,\eta)}),\overline{k}u_\gamma(W_T^{*,(t_0,w,\eta)})\big).$$

The proof of Proposition 3.6 is delegated to Section A.9.

3.2 A numerical illustration

We provide a numerical illustration of the optimal consumption-investment problem under the unit EIS Epstein–Zin SDU. Suppose that

$$\alpha(\eta) := \overline{\kappa}\big(\overline{\theta} - I_+(\eta)\big), \qquad \beta(\eta) := \overline{\beta}\sqrt{I_+(\eta)}, \quad \lambda(\eta) := \overline{\lambda} I_+(\eta),$$
$$r(\eta) := \overline{r}, \qquad m(\eta) := \overline{r} + \overline{m} I_+(\eta), \qquad \sigma(\eta) = \sqrt{I_+(\eta)},$$

where $I_+(\eta)$ is a $C_b^\infty(\mathbb{R})$- function satisfying

$$I_+(\eta) = \eta, \quad \text{if } \eta \in \left[\frac{1}{K}, K\right], \qquad I_+(\eta) \geq \frac{1}{K+1}, \quad \text{if } \eta < \frac{1}{K}, \qquad I_+(\eta) \leq K+1, \quad \text{if } \eta > K.$$

$\overline{\kappa} \in (0,\infty)$, $\overline{\theta} \in (0,\infty)$, $\overline{\beta} \in (0,\infty)$, $\overline{\lambda} \in (0,\infty)$, $\overline{r} \in (0,\infty)$, $\overline{m} \in \mathbb{R}$, and $K \in (0,\infty)$ are constants. Thus, η_t is a truncated CIR process with K chosen sufficiently large so that $I_+(\eta) = \eta$ throughout the computational region.

The jump size measure ζ is specified as the size distribution of negative returns below -3σ, based on the sample mean and sample variance estimated from 50 years of weekly US market portfolio returns since June 27, 1975. The data are obtained from Kenneth French's website. The threshold corresponding to a -3σ return is -0.0660 (-6.6%, weekly). Negative returns below -3σ were observed 23 times out of 2610 weeks (0.8812%). The theoretical probability of a return below -3σ under the normal distribution is 0.1350%. Therefore, "crashes" occur more frequently in the observed data than the normal distribution would suggest. Table 1 summarizes the jump-size distribution, showing the conditional probability that the jump magnitude falls within each specified bin. We model ζ using the discrete distribution reported in Table 1. For computations, the midpoint of each bin is used as the representative jump size. The resulting conditional mean jump size is -9.3152%. Note that time is measured in years. Weekly observations were utilized for the estimation of the empirical distribution of jump size, where the frequency was annualized.

Table 1: Jump Size Distribution

Bin (%)	Probability	Counts	Midpoints of Jump Size (%)
$[-7.50, -6.60)$	0.4348	10	-7.05
$[-10.00, -7.50)$	0.3043	7	-8.75
$[-12.50, -10.00)$	0.0870	2	-11.25
$[-15.00, -12.50)$	0.1304	3	-13.75
$[-17.50, -15.00)$	0.0000	0	-16.25
$[-20.00, -17.50)$	0.0435	1	-18.75

The parameter $\overline{\lambda}$ is set as follows. If the process is not truncated, the steady-state jump intensity is $\mathbb{E}[\lambda(\eta_t)] = \overline{\lambda}\mathbb{E}[\eta_t]$. In addition, the steady-state mean of η_t is $\overline{\theta}$ if $2\kappa\overline{\theta} \geq \overline{\beta}^2$. We therefore match $\mathbb{E}[\lambda(\eta_t)] = \overline{\lambda\theta}$ to the observed annual frequency 0.46 (23 events over 50 years in the data). Thus, $\overline{\lambda} = 0.46/\overline{\theta}$. Since the threshold constant K is chosen large so that $I_+(\eta) = \eta$ on the computational domain, this calibration is well approximated.

Under this specification, the first-order condition for the portfolio (3.11) becomes

$$I_+(\eta)\left(\overline{m} - \gamma\Pi^* + \overline{\lambda}\int_{\mathbb{R}}(1+\Pi^* x)^{-\gamma} x\zeta(\mathrm{d}x)\right) = 0, \quad \eta \in \mathbb{R}.$$

Hence, the optimal portfolio Π^* is a constant. The other parameters are set as follows.

$$\overline{r} = 0.0453, \qquad \overline{m} = 2.9864, \qquad \overline{\kappa} = 5, \qquad \overline{\theta} = 0.0270, \qquad \overline{\beta} = 0.25,$$
$$\overline{\lambda} = 0.46/\overline{\theta} \approx 17.01317, \qquad T = 10, \qquad \overline{k} = 1, \qquad \gamma = 5, \qquad \delta = 0.05.$$

The interest rate $\overline{r}$ is set to the sample annualized mean of the weekly risk-free rate over the same 50-year period (50 years since June 27, 1975). The coefficient of risk premium $\overline{m}$ and the steady-state volatility level $\overline{\theta}$ are calibrated to match the sample annualized mean and variance of market portfolio returns, using the same dataset employed for estimating the jump measure. For the parameters $\overline{\kappa}$ and $\overline{\beta}$, we adopt the values from Kraft et al. (2017). Although their model specification differs from ours, we use these values as a baseline case for our numerical illustration, because our main interest is in the theoretical properties of the model.

To obtain the solution to the reduced HJB equation (3.10), we solve the iterative PDEs (3.12) by the Crank–Nicolson method. For numerical parameters, the number of time grids is 100, the number of η grids is 150, and the maximum and minimum values of η grids are 1.5 and 0.01, respectively. h^n are linearly extrapolated in η at the boundaries.

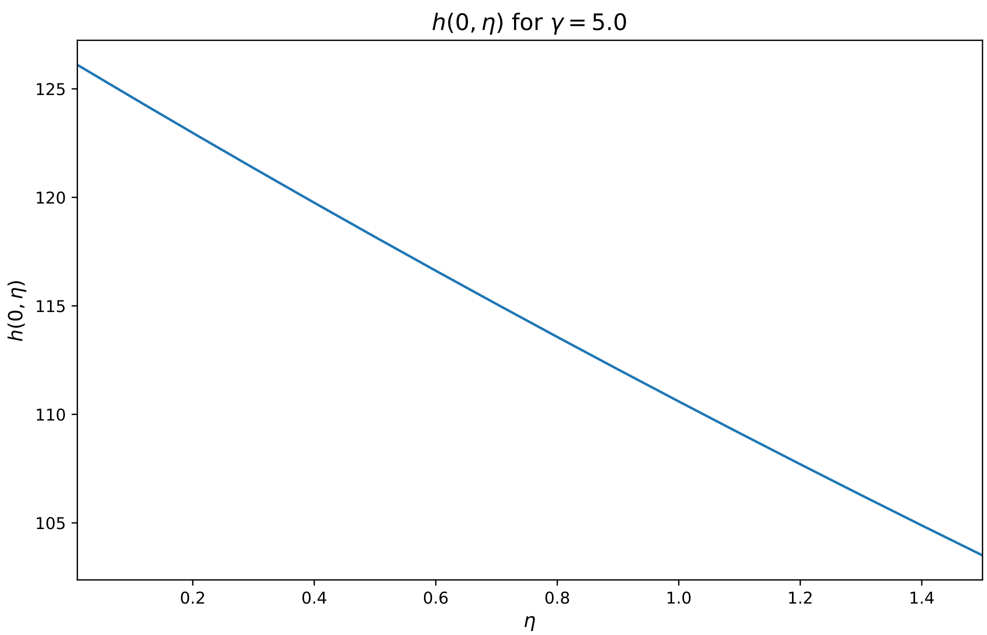

Figure 1: Plot of the PDE solution $h(0,\eta)$ for $\gamma = 5.0$.

Figure 1 displays the solution $h(0,\eta)$. The optimal portfolio with jumps is $\Pi^* = 0.2381$, while the optimal portfolio without jumps (i.e., $\lambda(\eta) = 0$) is $\overline{m}/\gamma = 0.5973$. This finding indicates that incorporating the risk of large negative shocks reduces the optimal allocation to the risky asset by approximately 36 percentage points.

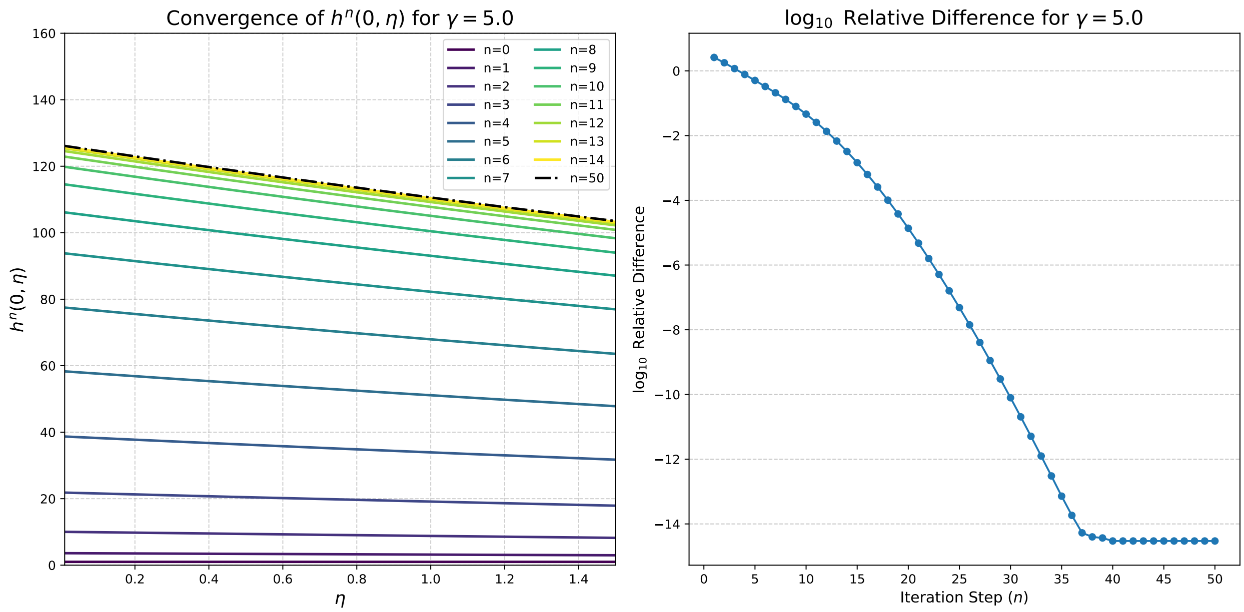

Figure 2: Convergence of the PDE solutions h^n for $\gamma = 5.0$.

Figure 2 illustrates the convergence process of the solutions to the iterative PDEs (3.12). In the case where $\gamma > 1$, $\delta \in (0,1)$, and $r(\eta) > 0$, $\widetilde{r}_{\text{stab.}}(\eta)$ is always negative if we choose $K_{\text{stab.}} = (1-\gamma)\delta(\log\delta - 1) > 0$, and we adopt this specification. The left panel of Figure 2 shows $h^n(0,\eta)$ at each iteration step n. The left panel suggests that the sequence $\{h^n\}_{n\in\mathbb{N}}$ converges monotonically to h. The right panel displays the convergence of the relative difference between iterations, defined as

$$\log_{10}\left(\max_{(t,\eta)}\left|\frac{h^n(t,\eta) - h^{n-1}(t,\eta)}{h^{n-1}(t,\eta)}\right|\right).$$

In this numerical example, the relative difference decreases to below 10^{-14} after the 37th iteration and then plateaus, indicating convergence up to a numerical noise floor. This result is consistent with Lemma 3.5.

Figure 3 illustrates the welfare cost of ignoring jump risk, measured by the certainty equivalent ratio (CE ratio). To define the CE ratio, we compare two scenarios. In the first, the agent follows the optimal strategy and obtains the value function $v(t,w,\eta) = h(t,\eta)u_\gamma(w)$. In the second, the agent uses the no-jump portfolio, $\Pi^{\text{nj}} := \overline{m}/\gamma$, in the presence of jumps. The consumption function remains the same: $C^{\text{nj}}(w) = C^*(w) = \delta w$. This suboptimal strategy yields the utility $v^{\text{nj}}(t,w,\eta) = h^{\text{nj}}(t,\eta)u_\gamma(w)$, where h^{nj} solves the corresponding PDE.

The CE ratio is the wealth multiplier $\chi(t,\eta)$ that makes the agent indifferent between the suboptimal strategy with initial wealth $\chi(t,\eta)w$ and the optimal strategy with initial wealth w, i.e., $v^{\text{nj}}(t,\chi(t,\eta)w,\eta) = v(t,w,\eta)$. This ratio at $t = 0$ is

$$\chi(0,\eta) = \left(\frac{h(0,\eta)}{h^{\text{nj}}(0,\eta)}\right)^{\frac{1}{1-\gamma}}.$$

Figure 3 plots this ratio. For $\gamma = 5$, the implied additional initial wealth requirement $\chi(0,\eta) - 1$

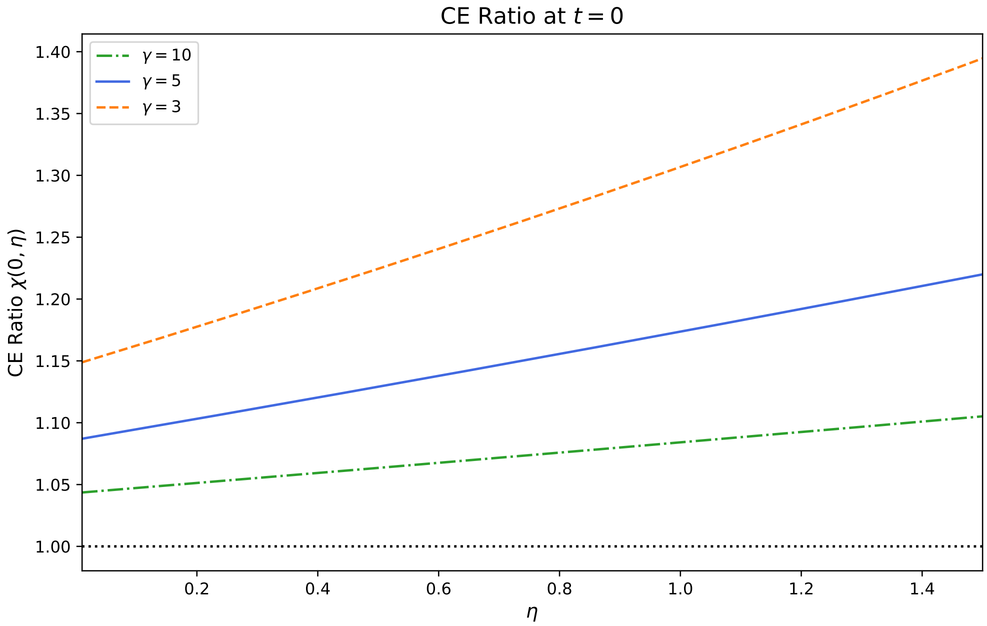

Figure 3: Certainty Equivalent Ratio at $t = 0$.

ranges from 8.7% to 22.0%. Thus, ignoring jump risk would require $\chi(0, \eta)$ times as much initial wealth to match the utility delivered by the optimal strategy. As shown in Figure 3, $\chi(0, \eta)$ decreases in γ, so the welfare loss is especially severe when γ is low. Remark 3.3 suggests that, in the approximation, the optimal portfolio Π^* also scales approximately as $1/\gamma$. Thus, the absolute wedge $\Pi^* - \Pi^{\mathrm{nj}}$ is of order $1/\gamma$, so the welfare loss (and hence $\chi - 1$) can be large when γ is small.

4 Concluding Remarks

In this paper, we examine the existence, uniqueness, and comparison of the unit EIS Epstein–Zin SDU in a Brownian-jump setting. We also consider an optimal consumption-investment problem under the unit EIS Epstein–Zin SDU for applications. In particular, our existence result is formulated on a consumption class that strictly enlarges the admissible class considered in Schroder and Skiadas (1999), whereas our comparison-based uniqueness is established on a different admissibility class tailored to the comparison principle. In this sense, our contribution for uniqueness is complementary to Schroder and Skiadas (1999). Applied asset pricing and macro-finance studies often assume milder conditions than Schroder and Skiadas (1999), so this study contributes to developing mathematical foundations for these applied settings. One possible extension is to relax the integrability conditions used for the uniqueness and comparison theorems. Indeed, the conditions for the comparison theorem are more restrictive than those for existence. Therefore, it is expected that these integrability conditions can be further relaxed. We leave a detailed examination of this direction for future studies.

A Omitted Proofs

In this section, we present proofs omitted in the main text. We introduce the following classes of stochastic processes and random fields for notational simplicity. Let $\mathbb{S}^2$ be the set of real-valued, right-continuous, and progressively measurable processes on $[0,T]$ such that $Y \in \mathbb{S}^2$ satisfies

$$\|Y\|_{\mathbb{S}^2} := \sqrt{\mathbb{E}\left[\sup_{t\in[0,T]} |Y_t|^2\right]} < \infty.$$

Let $\mathbb{S}^\infty$ be the set of real-valued, right-continuous, and progressively measurable processes on $[0,T]$ such that $Y \in \mathbb{S}^\infty$ satisfies

$$\|Y\|_{\mathbb{S}^\infty} := \operatorname*{ess\,sup}_{(t,\omega)\in[0,T]\times\Omega} |Y_t(\omega)| < \infty.$$

Let $\mathbb{H}^2$ be the set of $\mathbb{R}^d$-valued and progressively measurable processes on $[0,T]$ such that $Z \in \mathbb{H}^2$ satisfies

$$\|Z\|_{\mathbb{H}^2} := \sqrt{\mathbb{E}\left[\int_0^T \|Z_t\|^2 \mathrm{d}t\right]} < \infty.$$

Let $\mathbb{J}^2$ be the set of real-valued and predictable random fields on $[0,T]\times J$ such that $\psi \in \mathbb{J}^2$ satisfies

$$\|\psi\|_{\mathbb{J}^2} := \sqrt{\mathbb{E}\left[\int_0^T \int_J |\psi_t(x)|^2 \lambda_t(x)\zeta(\mathrm{d}x)\mathrm{d}t\right]} < \infty.$$

Let $\mathbb{J}^\infty$ be the set of real-valued and predictable random fields on $[0,T]\times J$ such that $\psi \in \mathbb{J}^\infty$ satisfies

$$\|\psi\|_{\mathbb{J}^\infty} := \operatorname*{ess\,sup}_{(t,\omega,x)\in[0,T]\times\Omega\times J} |\psi_t(\omega,x)| < \infty.$$

Let $\mathbb{L}$ be the set of real-valued measurable functions on $(J,\mathcal{J})$.

For convenience, we define a function from $\mathbb{R}$ to $\mathbb{R}$ such that

$$j_\alpha(x) := \frac{e^{\alpha x}-1}{\alpha} - x, \qquad x \in \mathbb{R},$$

where α is a non-zero constant. The behavior of j_α depends on the value of the parameter α. If $\alpha > 0$, then j_α is convex and $j_\alpha(x) \ge j_\alpha(0) = 0$ for any $x \in \mathbb{R}$. If $\alpha < 0$, then j_α is concave and $j_\alpha(x) \le j_\alpha(0) = 0$ for any $x \in \mathbb{R}$.

We also introduce the behavior of the unit EIS Epstein–Zin aggregator f. Seiferling and Seifried (2016) show that the general Epstein–Zin aggregator with RRA γ and EIS ς, denoted by $f^{gEZ}(c,v;\gamma,\varsigma)$, satisfies the following properties. If $\gamma\varsigma > 1$, then $f^{gEZ}(c,v;\gamma,\varsigma)$ is convex in v and concave in c. If $\gamma\varsigma < 1$, then $f^{gEZ}(c,v;\gamma,\varsigma)$ is jointly concave in c and v. Furthermore, it satisfies

$$f^{gEZ}(c,v;\gamma,\varsigma)\begin{cases} \ge \delta\big(u_\gamma(c)-v\big), & \text{if } \gamma\varsigma > 1, \\ \le \delta\big(u_\gamma(c)-v\big), & \text{if } \gamma\varsigma < 1. \end{cases}$$

Since $f^{gEZ}(c,v;\gamma,\varsigma) \to f(c,v)$ as $\varsigma \to 1$, we obtain the following results as an immediate corollary of Seiferling and Seifried (2016). In the case of $\gamma > 1$, $f(c,v)$ is convex in v and concave in c, and $f(c,v) \ge \delta(u_\gamma(c)-v)$ for any $(c,v) \in (0,\infty)\times(-\infty,0)$. In the case of $\gamma < 1$, $f(c,v)$ is jointly concave in c and v, and $f(c,v) \le \delta(u_\gamma(c) - v)$ for any $(c,v) \in (0,\infty) \times (0,\infty)$. These properties play a crucial role in proving the existence and comparison theorem for the unit EIS Epstein–Zin SDU.

A.1 Proof of Proposition 2.4

Proof. We only show the case of $\gamma > 1$, and the case of $\gamma < 1$ can be proved similarly. We first assume bounded C and ξ (i.e., $(C,\xi) \in \mathcal{C}_T^{\mathrm{B}}$), and then consider a general case. To show existence, four steps are necessary.

1. Lipschitz approximations and global boundaries. First suppose $(C,\xi) \in \mathcal{C}_T^{\mathrm{B}}$. For any $n \in \mathbb{N}$, let $\rho_n : \mathbb{R} \to [0,1]$ be a continuously differentiable function such that $\rho_n(x) = 1$ if $|x| \le n$ and $\rho_n(x) = 0$ if $|x| > n+1$. Further suppose that $\rho_n(x) \le \rho_{n+1}(x)$ holds for any $x \in \mathbb{R}$ and $n \in \mathbb{N}$ and that the absolute value of its derivative, $|\rho_n'(x)|$, is bounded uniformly in $x \in \mathbb{R}$ and $n \in \mathbb{N}$. For any $n \in \mathbb{N}$, consider a BSDE with respect to (Y^n, Z^n, ψ^n) on $[0,T]$ such that

$$\begin{aligned} Y_t^n &= \xi^Y + \int_t^T \big(\delta(\log C_s - Y_s^n) + h_n(s, Z_s^n, \psi_s^n)\big)\mathrm{d}s - \int_t^T (Z_s^n)^\top \mathrm{d}B_s - \int_t^T \int_J \psi_s^n(x)\widetilde{\mu}(\mathrm{d}s, \mathrm{d}x), \\ h_n(s,z,\phi) &:= \frac{1-\gamma}{2}\|z\|^2 \rho_n(\|z\|^2) + \int_J j_{1-\gamma}^n\big(\phi(x)\big)\lambda_s(x)\zeta(\mathrm{d}x), \qquad (z,\phi) \in \mathbb{R}^d \times \mathbb{L}, \end{aligned} \tag{A.1}$$

where $\xi^Y := \log(u_\gamma^{-1}(\xi))$ and $j_{1-\gamma}^n$ is defined as

$$j_{1-\gamma}^n(x) = \begin{cases} j_{1-\gamma}(n) + j_{1-\gamma}'(n)(x-n), & x > n, \\ j_{1-\gamma}(x), & x \in [-n, n], \\ j_{1-\gamma}(-n) + j_{1-\gamma}'(-n)(x+n), & x < -n. \end{cases}$$

By the definitions of ρ_n and $j_{1-\gamma}^n$ and the finite activity of $\nu = \lambda\zeta$, $h_n(s,z,\phi)$ is Lipschitz in (z,ϕ) in the sense that there exists a non-negative constant K_n such that

$$|h_n(s,z,\phi) - h_n(s,\widetilde{z},\widetilde{\phi})| \le K_n\left(\|z - \widetilde{z}\| + \sqrt{\int_J |\phi(x) - \widetilde{\phi}(x)|^2 \lambda_s(x)\zeta(\mathrm{d}x)}\right),$$

for any $(z,\phi), (\widetilde{z},\widetilde{\phi}) \in \mathbb{R}^d \times \mathbb{L}$ and $s \in [0,T]$. Moreover, ξ^Y and $(\log C_t)_{t\in[0,T]}$ are bounded since $(C,\xi) \in \mathcal{C}_T^{\mathrm{B}}$. Therefore, the Lipschitz BSDE (A.1) has a unique solution $(Y^n, Z^n, \psi^n) \in \mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{J}^2$ (for example, see Theorem 3.1.1 in Delong (2013)). The Lipschitz property of the generator allows us to derive a standard a priori estimate for Y^n. Applying the generalized Ito formula to $e^{ct}|Y_t^n|^2$ for an appropriate constant c and taking conditional expectations leads to

$$|Y_t^n|^2 \le K_n \mathbb{E}_t\left[|\xi^Y|^2 + \int_t^T |\log C_s|^2 \mathrm{d}s\right] \le K_n\Big(|\overline{\xi}^Y|^2 + \|\log C\|_{\mathbb{S}^\infty}^2 T\Big) < \infty,$$

for any $t \in [0,T]$, where K_n is a constant depending on n, and $\overline{\xi}^Y$ is a finite positive constant satisfying $|\xi^Y| \le \overline{\xi}^Y$ a.s. This implies $Y^n \in \mathbb{S}^\infty$. However, this boundary depends on n, so we examine global bounds of Y^n for all $n \in \mathbb{N}$. We first examine an upper global bound. Consider a

BSDE for $(Y^{\log}, Z^{\log}, \psi^{\log})$ on $[0,T]$ such that

$$Y_t^{\log} = \xi^Y + \int_t^T \delta(\log C_s - Y_s^{\log})\mathrm{d}s - \int_t^T (Z_s^{\log})^\top \mathrm{d}B_s - \int_t^T \int_J \psi_s^{\log}(x)\widetilde{\mu}(\mathrm{d}s, \mathrm{d}x).$$

This BSDE is Lipschitz, so it has a unique solution in $\mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{J}^2$ such that

$$Y_t^{\log} = \mathbb{E}_t \left[\int_t^T \delta e^{-\delta(s-t)} \log C_s \mathrm{d}s + e^{-\delta(T-t)}\xi^Y \right].$$

Thus, $\|Y^{\log}\|_{\mathbb{S}^\infty} < \infty$. Let $\Delta^{\log} Y_s := Y_s^{\log} - Y_s^n$, $\Delta^{\log} Z_s := Z_s^{\log} - Z_s^n$, and $\Delta^{\log}\psi_s := \psi_s^{\log} - \psi_s^n$. Then, we have

$$\Delta^{\log} Y_t = \int_t^T \Big(-\delta\Delta^{\log} Y_s - h_n(s, Z_s^n, \psi_s^n)\Big)\mathrm{d}s - \int_t^T (\Delta^{\log} Z_s)^\top \mathrm{d}B_s - \int_t^T \int_J \Delta^{\log}\psi_s(x)\widetilde{\mu}(\mathrm{d}s, \mathrm{d}x)$$

Since $\gamma > 1$ implies $1-\gamma < 0$ and $j_{1-\gamma}^n \leq 0$, we have $h_n \leq 0$. Accordingly,

$$\Delta^{\log} Y_t = \mathbb{E}_t \left[-\int_t^T e^{-\delta(s-t)} h_n(s, Z_s^n, \psi_s^n)\mathrm{d}s \right] \geq 0,$$

for any $t \in [0,T]$. Therefore, $Y_t^n \leq Y_t^{\log} \leq \|Y^{\log}\|_{\mathbb{S}^\infty} < \infty$ holds for any $t \in [0,T]$ a.s.

We here examine a lower global bound. Let $U_t^n := u_\gamma(\exp\{Y_t^n\})$. Note that $U_t^n < 0$ holds for $\gamma > 1$ by definition. Then, the generalized Ito formula yields

$$\begin{aligned} U_t^n &= \xi + \int_t^T \Big(f(C_s, U_s^n) + e^{(1-\gamma)Y_s^n}\big(h_n(s, Z_s^n, \psi_s^n) - h(s, Z_s^n, \psi_s^n)\big)\Big)\mathrm{d}s \\ &\quad - \int_t^T (Z_s^{n,U})^\top \mathrm{d}B_s - \int_t^T \int_J \psi_s^{n,U}(x)\widetilde{\mu}(\mathrm{d}s, \mathrm{d}x), \\ h(s,z,\phi) &:= \frac{1-\gamma}{2}\|z\|^2 + \int_J j_{1-\gamma}\big(\phi(x)\big)\lambda_s(x)\zeta(\mathrm{d}x), \qquad (z,\phi) \in \mathbb{R}^d \times \mathbb{L}, \\ Z_s^{n,U} &:= e^{(1-\gamma)Y_{s-}^n} Z_s^n, \quad \psi_s^{n,U}(x) := \frac{e^{(1-\gamma)Y_{s-}^n}}{1-\gamma}\Big(e^{(1-\gamma)\psi_s^n(x)} - 1\Big), \end{aligned}$$

for any $t \in [0,T]$. Since $\Delta Y_t^n = \int_J \psi_t^n(x)\mu(\{t\}, \mathrm{d}x)$ and Y^n is bounded, we have $\|\psi^n\|_{\mathbb{J}^\infty} = \|\Delta Y_\cdot^n\|_{\mathbb{S}^\infty} \leq 2\|Y^n\|_{\mathbb{S}^\infty}$ $\mathbb{P}\otimes\nu$-a.e. Accordingly, it holds that $\|Z^{n,U}\|_{\mathbb{H}^2} + \|\psi^{n,U}\|_{\mathbb{J}^2} < \infty$. Meanwhile, consider another BSDE for $(U^{\mathrm{CRRA}}, Z^{\mathrm{CRRA}}, \psi^{\mathrm{CRRA}})$ on $[0,T]$ such that

$$U_t^{\mathrm{CRRA}} = \xi + \int_t^T \delta\Big(u_\gamma(C_s) - U_s^{\mathrm{CRRA}}\Big)\mathrm{d}s - \int_t^T (Z_s^{\mathrm{CRRA}})^\top \mathrm{d}B_s - \int_t^T \int_J \psi_s^{\mathrm{CRRA}}(x)\widetilde{\mu}(\mathrm{d}s, \mathrm{d}x).$$

This BSDE is Lipschitz, so it has a unique solution in $\mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{J}^2$ such that

$$U_t^{\mathrm{CRRA}} = \mathbb{E}_t \left[\int_t^T \delta e^{-\delta(s-t)} u_\gamma(C_s)\mathrm{d}s + e^{-\delta(T-t)}\xi \right].$$

Thus, $\|U^{\mathrm{CRRA}}\|_{\mathbb{S}^\infty} < \infty$. Let $\Delta^{\mathrm{CRRA}} U_s := U_s^{\mathrm{CRRA}} - U_s^n$, $\Delta^{\mathrm{CRRA}} Z_s := Z_s^{\mathrm{CRRA}} - Z_s^{n,U}$, and

$\Delta^{\text{CRRA}}\psi_s := \psi_s^{\text{CRRA}} - \psi_s^{n,U}$. Then we have

$$\Delta^{\text{CRRA}}U_t = \int_t^T \left[\delta\Big(u_\gamma(C_s) - U_s^{\text{CRRA}}\Big) - f(C_s, U_s^n) - e^{(1-\gamma)Y_s^n}\Big(h_n(s, Z_s^n, \psi_s^n) - h(s, Z_s^n, \psi_s^n)\Big)\right]\mathrm{d}s$$
$$- \int_t^T (\Delta^{\text{CRRA}}Z_s)^\top \mathrm{d}B_s - \int_t^T \int_J \Delta^{\text{CRRA}}\psi_s(x)\widetilde{\mu}(\mathrm{d}s, \mathrm{d}x).$$

Note that $\delta(u_\gamma(c)-v)-f(c,v) \le 0$ for any $c \in (0,\infty)$ and $v < 0$, and that $h_n(s,z,\phi)-h(s,z,\phi) \ge 0$ for any $(s,z,\phi) \in [0,T]\times\mathbb{R}^d\times\mathbb{L}$ because $j_{1-\gamma}^n \ge j_{1-\gamma}$. Hence, we have

$$\Delta^{\text{CRRA}}U_t \le -\delta\int_t^T \Delta^{\text{CRRA}}U_s\mathrm{d}s - \int_t^T (\Delta^{\text{CRRA}}Z_s)^\top \mathrm{d}B_s - \int_t^T \int_J \Delta^{\text{CRRA}}\psi_s(x)\widetilde{\mu}(\mathrm{d}s, \mathrm{d}x).$$

Therefore, applying the integral-by-parts formula and taking the conditional expectation yields $\Delta^{\text{CRRA}}U_t \le 0$ for any $t \in [0,T]$ a.s. Thus,

$$Y_t^n = \log\circ u_\gamma^{-1}(U_t^n) \ge \log\circ u_\gamma^{-1}(U_t^{\text{CRRA}}) \ge \log\circ u_\gamma^{-1}(-\|U^{\text{CRRA}}\|_{\mathbb{S}^\infty}) > -\infty$$

holds for any $t \in [0,T]$ a.s. In summary, we obtain the global bound of Y^n: it holds that

$$-\infty < \log\circ u_\gamma^{-1}(-\|U^{\text{CRRA}}\|_{\mathbb{S}^\infty}) \le \log\circ u_\gamma^{-1}(U_t^{\text{CRRA}}) \le Y_t^n \le Y_t^{\log} \le \|Y^{\log}\|_{\mathbb{S}^\infty} < \infty, \qquad \text{(A.2)}$$

for any $t \in [0,T]$ and $n \in \mathbb{N}$. To obtain the global bounds of Z^n and ψ^n, we check the following inequality.

$$-\delta|\log C_t| - \delta|y| - \frac{\gamma-1}{2}\|z\|^2 - \int_J j_{\gamma-1}(-\phi(x))\lambda_t(x)\zeta(\mathrm{d}x)$$
$$\le \delta\big(\log C_t - y\big) + h_n(t,z,\phi) \le \delta|\log C_t| + \delta|y| + \frac{\gamma-1}{2}\|z\|^2 + \int_J j_{\gamma-1}(\phi(x))\lambda_t(x)\zeta(\mathrm{d}x), \qquad \text{(A.3)}$$

for all $(t,y,z,\phi) \in [0,T]\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{L}$ a.s. We use the identity $j_{1-\gamma}(\phi) = -j_{\gamma-1}(-\phi)$ to obtain the bounds in (A.3). Therefore, Lemma 3.1 in Fujii and Takahashi (2018) implies

$$\mathbb{E}\left[\int_0^T \|Z_t^n\|^2\mathrm{d}t + \int_0^T\int_J |\psi_t^n(x)|^2\lambda_t(x)\zeta(\mathrm{d}x)\mathrm{d}t\right]$$
$$\le \frac{3e^{4(\gamma-1)\|Y^n\|_{\mathbb{S}^\infty}}}{(\gamma-1)^2}\Big(1 + 2(\gamma-1)T\big(\delta\|Y^n\|_{\mathbb{S}^\infty} + \|\log C\|_{\mathbb{S}^\infty}\big)\Big).$$

The right-hand side of the above inequality is bounded uniformly in n, so (Z^n, ψ^n) is uniformly bounded in $\mathbb{H}^2\times\mathbb{J}^2$.

2. Monotone convergence of Y^n and existence of Y. Let

$$g_n(t,y,z,\phi) := \delta(\log C_t - y) + h_n(t,z,\phi), \qquad (t,y,z,\phi) \in [0,T]\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{L}.$$

g_n is the Lipschitz generator of the BSDE (A.1). Furthermore, the inequality (A.3) holds, and the so-called A_Γ condition also holds: for any $\phi, \widetilde{\phi} \in \mathbb{L}$ and $t \in [0,T]$, if $|\phi(x)| \vee |\widetilde{\phi}(x)| \le K$ holds

$\nu(\{t\},\cdot)$-a.e. for a constant $K>0$, then

$$g_n(t,y,z,\phi)-g_n(t,y,z,\widetilde{\phi})\leq\int_J\Gamma_t^{\phi,\widetilde{\phi}}(x)\big(\phi(x)-\widetilde{\phi}(x)\big)\lambda_t(x)\zeta(\mathrm{d}x),\tag{A.4}$$

for any $(y,z)\in\mathbb{R}\times\mathbb{R}^d$ and $n\in\mathbb{N}$, where $\Gamma^{\phi,\widetilde{\phi}}$ is a random field depending only on $(\phi,\widetilde{\phi})$ and satisfying $C_K^1\leq\Gamma_t^{\phi,\widetilde{\phi}}(x)\leq C_K^2$ for all $(t,x)\in[0,T]\times J$ in which $C_K^1>-1$ and $C_K^2>0$ are constants depending on K. The A_Γ-condition (A.4) is a key property to apply the comparison theorem for BSDEs with jumps. Morlais (2010) shows that the A_Γ-condition (A.4) holds in a setting similar to this paper. To observe (A.4), we have

$$g_n(t,y,z,\phi)-g_n(t,y,z,\widetilde{\phi})=\int_J\bigg(\int_0^1\frac{\mathrm{d}j_{1-\gamma}^n(w)}{\mathrm{d}w}\bigg|_{w=k\phi(x)+(1-k)\widetilde{\phi}(x)}\mathrm{d}k\bigg)\big(\phi(x)-\widetilde{\phi}(x)\big)\lambda_t(x)\zeta(\mathrm{d}x),$$

where we have used the fundamental theorem of calculus. Then, for any $|w|\leq K$, we have

$$-1<e^{(1-\gamma)K}-1\leq\frac{\mathrm{d}j_{1-\gamma}^n(w)}{\mathrm{d}w}\leq e^{(\gamma-1)K}-1<\infty.$$

Hence, let

$$\Gamma_t^{n,\phi,\widetilde{\phi}}(x):=\int_0^1\frac{\mathrm{d}j_{1-\gamma}^n(w)}{\mathrm{d}w}\bigg|_{w=k\phi(x)+(1-k)\widetilde{\phi}(x)}\mathrm{d}k,$$

and then $e^{(1-\gamma)K}-1\leq\Gamma_t^{n,\phi,\widetilde{\phi}}\leq e^{(\gamma-1)K}-1$ holds $\nu(\{t\},\cdot)$-a.e. for any $(t,\phi,\widetilde{\phi})\in[0,T]\times\mathbb{L}\times\mathbb{L}$ with $|\phi(x)|\vee|\widetilde{\phi}(x)|\leq K$, $\nu(\{t\},\cdot)$-a.e. Moreover, by the definition of $j_{1-\gamma}^n$, $j_{1-\gamma}^n(\phi(x))=j_{1-\gamma}(\phi(x))$ holds for any $\phi\in\mathbb{L}$ and $n\in\mathbb{N}$ with $|\phi|\leq n$. Accordingly, $\Gamma_t^{n,\phi,\widetilde{\phi}}=\Gamma_t^{m,\phi,\widetilde{\phi}}$ holds $\nu(\{t\},\cdot)$-a.e. for any $(t,\phi,\widetilde{\phi})\in[0,T]\times\mathbb{L}\times\mathbb{L}$ and $n,m\in\mathbb{N}$ with $n\leq m$ and $|\phi(x)|\vee|\widetilde{\phi}(x)|\leq n$, $\nu(\{t\},\cdot)$-a.e. Thus, the A_Γ condition (A.4) holds, in particular, uniformly in sufficiently large $n\in\mathbb{N}$. Furthermore, by the nonincreasing property of ρ_n and $j_{1-\gamma}^n$ in n, $h_{n+1}(t,y,z,\phi)\leq h_n(t,y,z,\phi)$ holds for any $(t,y,z,\phi)\in[0,T]\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{L}$ and $n\in\mathbb{N}$ a.s. This implies that $g_{n+1}(t,y,z,\phi)\leq g_n(t,y,z,\phi)$ also holds a.s. Thus, the comparison theorem for Lipschitz BSDEs (see Theorem 3.2.2 in Delong (2013)) implies $Y_t^{n+1}\leq Y_t^n$ for any $t\in[0,T]$ and $n\in\mathbb{N}$.

By Step 1, $\{Y^n\}_{n\in\mathbb{N}}$ is uniformly bounded in $\mathbb{S}^\infty$. Moreover, $\{g_n\}$ satisfies the uniform quadratic–exponential growth bound (A.3), the uniform A_Γ condition for sufficiently large n[2], and $g_n\downarrow g$ pointwise as $n\to\infty$. Hence, Proposition 4.1 in Fujii and Takahashi (2018) yields the existence of a solution $(Y,Z,\psi)\in\mathbb{S}^\infty\times\mathbb{H}^2\times\mathbb{J}^2$ to (2.6), and $(Y^n,Z^n,\psi^n)\to(Y,Z,\psi)$ in $\mathbb{S}^\infty\times\mathbb{H}^2\times\mathbb{J}^2$ as $n\to\infty$. Note that $\|\psi\|_{\mathbb{J}^\infty}\leq 2\|Y\|_{\mathbb{S}^\infty}$ holds, so $\psi\in\mathbb{J}^\infty$.

3. Representation as the utility. Let $U_t:=u_\gamma(\exp\{Y_t\})$ for $t\in[0,T]$. Applying the generalized Ito formula to U yields

$$U_t=\xi+\int_t^T f(C_s,U_s)\mathrm{d}s-\int_t^T e^{(1-\gamma)Y_s}Z_s^\top\mathrm{d}B_s-\int_t^T\int_J u_\gamma(\exp\{Y_{s-}\})\Big(e^{(1-\gamma)\psi_s(x)}-1\Big)\widetilde{\mu}(\mathrm{d}s,\mathrm{d}x),$$

[2]The A_Γ-condition in Fujii and Takahashi (2018) is stated for general jump measures (possibly of infinite activity), where the factor $1\wedge|x|$ is used to control small jumps. In our finite-activity setting, $\nu(\mathrm{d}t,\mathrm{d}x)=\lambda_t(x)\zeta(\mathrm{d}x)\mathrm{d}t$ has finite mass on $[0,T]\times J$. Hence, small-jump damping $1\wedge|x|$ in Fujii and Takahashi (2018) is not needed. It suffices that for bounded ϕ and $\widetilde{\phi}$, the field $\Gamma^{n,\phi,\widetilde{\phi}}$ constructed above is predictable, bounded and $\Gamma_t^{n,\phi,\widetilde{\phi}}>-1$ ν-a.e.

for $t \in [0,T]$. Since (Y, Z, ψ) is in $\mathbb{S}^\infty \times \mathbb{H}^2 \times \mathbb{J}^\infty$, it follows that $e^{(1-\gamma)Y} Z \in \mathbb{H}^2$ and $u_\gamma(\exp\{Y_{\cdot-}\})(e^{(1-\gamma)\psi} - 1)) \in \mathbb{J}^2$. Therefore, the stochastic integrals are square-integrable martingales. Taking the conditional expectation, we have

$$U_t = \mathbb{E}_t \left[\xi + \int_t^T f(C_s, U_s) \mathrm{d}s \right].$$

Furthermore, by (A.2), U is bounded, and hence it is uniformly integrable. Moreover, $(1-\gamma)U_t = \exp\{(1-\gamma)Y_t\} > 0$ holds a.s. for all $t \in [0,T]$. Thus, U is a unit EIS Epstein–Zin SDU of (C, ξ).

4. Unbounded (C, ξ). Finally, we consider an unbounded (C, ξ). Fix $(C, \xi) \in \mathcal{C}^{\mathrm{E}}_{\gamma,T}$ arbitrarily. For any $m, n \in \mathbb{N}$, let $(C^{m,n}, \xi^{m,n})$ be a cut-off version of (C, ξ) such that

$$C^{m,n}_t := (C_t \vee 1/m) \wedge n, \quad \xi^{m,n} := u_\gamma \Big(\big(u_\gamma^{-1}(\xi) \vee (1/m)\big) \wedge n \Big),$$

for $t \in [0,T]$. Then, $(C^{m,n}, \xi^{m,n})$ is bounded and in $\mathcal{C}^{\mathrm{B}}_T$. From the arguments in Steps 1 to 3, there exists a unit EIS Epstein–Zin SDU of $(C^{m,n}, \xi^{m,n})$ such that

$$U^{m,n}_t = \mathbb{E}_t \left[\xi^{m,n} + \int_t^T f(C^{m,n}_s, U^{m,n}_s) \mathrm{d}s \right].$$

We next study a dominating stochastic process for the sequence $\{f(C^{m,n}_\cdot, U^{m,n}_\cdot)\}_{m,n}$, which will allow us to pass to the limits $n \to \infty$ and $m \to \infty$ in the conditional expectation representation. Note that $U^{m,n}$ satisfies the global bound such that

$$|U^{m,n}_t| \le \left| U^{\mathrm{CRRA}}_t (C^{m,n}, \xi^{m,n}) \right| \le \mathbb{E}_t \left[\int_0^T \delta \frac{(C_s \vee 1)^{1-\gamma}}{|1-\gamma|} \mathrm{d}s + \frac{\big((1-\gamma)\xi\big) \vee 1}{|1-\gamma|} \right] < \infty.$$

By the definition of f, we have

$$f(C^{m,n}_t, U^{m,n}_t) = \delta(1-\gamma) U^{m,n}_t \log \left(\frac{C^{m,n}_t}{((1-\gamma)U^{m,n}_t)^{1/(1-\gamma)}} \right) = -\delta u_\gamma(C^{m,n}_t) \frac{U^{m,n}_t}{u_\gamma(C^{m,n}_t)} \log \left(\frac{U^{m,n}_t}{u_\gamma(C^{m,n}_t)} \right),$$

for $t \in [0,T]$. Since $U^{m,n}_t / u_\gamma(C^{m,n}_t) \le U^{\mathrm{CRRA}}_t(C^{m,n}, \xi^{m,n}) / u_\gamma(C^{m,n}_t)$ holds for any $t \in [0,T]$, we examine the global bound of $U^{\mathrm{CRRA}}_t(C^{m,n}, \xi^{m,n}) / u_\gamma(C^{m,n}_t)$. Let

$$\mathcal{D}^{m,n}_t := \frac{U^{\mathrm{CRRA}}_t(C^{m,n}, \xi^{m,n})}{u_\gamma(C^{m,n}_t)} = \mathbb{E}_t \left[\int_t^T \delta e^{-\delta(s-t)} \left(\frac{C^{m,n}_s}{C^{m,n}_t} \right)^{1-\gamma} \mathrm{d}s + e^{-\delta(T-t)} \frac{(1-\gamma)\xi^{m,n}}{(C^{m,n}_t)^{1-\gamma}} \right],$$

for $t \in [0,T]$. Note that since $C \in \mathcal{C}^{\mathrm{E}}_{\gamma,T}$, from (2.9), there exists a non-negative and progressively measurable process $K^{(C,\xi)}$ such that

$$\mathbb{E}_t \left[\int_t^T \delta e^{-\delta(s-t)} C_s^{1-\gamma} \mathrm{d}s + e^{-\delta(T-t)} (1-\gamma)\xi \right] \le K^{(C,\xi)}_t C_t^{1-\gamma}$$

for any $t \in [0,T]$ a.s. We shall show that $\mathcal{D}^{m,n}_t$ is dominated by $K^{(C,\xi)}_t + 1$, $\mathrm{d}t \times \mathrm{d}\mathbb{P}$-a.e., uniformly in m and n. Let $\Omega^* \in \mathcal{F}$ be an almost-sure set in which the above inequality is satisfied for any rational number $t \in [0,T]$. Then, $\mathbb{P}(\Omega^*) = 1$. Fix a rational number $t \in [0,T]$ and $\omega \in \Omega^*$ arbitrarily. We shall consider three cases. (1) The case $C_t(\omega) < 1/m$: Then, $(C^{m,n}_t(\omega))^{1-\gamma} = m^{-(1-\gamma)}$. Meanwhile,

$(C_s^{m,n})^{1-\gamma} \leq m^{-(1-\gamma)}$ and $(1-\gamma)\xi^{m,n} \leq m^{-(1-\gamma)}$. Hence,

$$\mathcal{D}_t^{m,n}(\omega) \leq \frac{1}{m^{-(1-\gamma)}} \mathbb{E}_t \left[\int_t^T \delta e^{-\delta(s-t)} m^{-(1-\gamma)} \mathrm{d}s + e^{-\delta(T-t)} m^{-(1-\gamma)} \right](\omega) = 1.$$

(2) The case $1/m \leq C_t(\omega) < n$: Then, $(C_t^{m,n}(\omega))^{1-\gamma} = (C_t(\omega))^{1-\gamma} \geq n^{1-\gamma}$. For any $s \in (t,T]$, we have

$$\begin{aligned}(C_s^{m,n})^{1-\gamma} &= \left(C_s^{1-\gamma} \wedge m^{-(1-\gamma)} \right) \vee n^{1-\gamma} \leq C_s^{1-\gamma} \vee n^{1-\gamma} \leq C_s^{1-\gamma} + n^{1-\gamma}, \\ (1-\gamma)\xi^{m,n} &= \left(\left((1-\gamma)\xi\right) \wedge m^{-(1-\gamma)} \right) \vee n^{1-\gamma} \leq (1-\gamma)\xi + n^{1-\gamma}.\end{aligned}$$

Hence,

$$\begin{aligned}\mathcal{D}_t^{m,n}(\omega) &\leq \frac{1}{(C_t^{m,n}(\omega))^{1-\gamma}} \mathbb{E}_t \left[\int_t^T \delta e^{-\delta(s-t)} \left[C_s^{1-\gamma} + n^{1-\gamma} \right] \mathrm{d}s + e^{-\delta(T-t)} \left[(1-\gamma)\xi + n^{1-\gamma} \right] \right](\omega) \\ &= \frac{1}{(C_t^{m,n}(\omega))^{1-\gamma}} \mathbb{E}_t \left[\int_t^T \delta e^{-\delta(s-t)} C_s^{1-\gamma} \mathrm{d}s + e^{-\delta(T-t)} (1-\gamma)\xi \right](\omega) \\ &\quad + \frac{1}{(C_t^{m,n}(\omega))^{1-\gamma}} \mathbb{E}_t \left[\int_t^T \delta e^{-\delta(s-t)} n^{1-\gamma} \mathrm{d}s + e^{-\delta(T-t)} n^{1-\gamma} \right](\omega) \\ &\leq \frac{K_t^{(C,\xi)}(\omega)(C_t(\omega))^{1-\gamma}}{(C_t^{m,n}(\omega))^{1-\gamma}} + \frac{n^{1-\gamma}}{(C_t^{m,n}(\omega))^{1-\gamma}} \leq \frac{K_t^{(C,\xi)}(\omega)(C_t(\omega))^{1-\gamma}}{(C_t(\omega))^{1-\gamma}} + \frac{n^{1-\gamma}}{n^{1-\gamma}} = K_t^{(C,\xi)}(\omega) + 1.\end{aligned}$$

(3) The case $C_t(\omega) \geq n$: Then, $(C_t(\omega))^{1-\gamma} \leq n^{1-\gamma} = (C_t^{m,n}(\omega))^{1-\gamma}$, and hence,

$$\mathcal{D}_t^{m,n}(\omega) \leq \frac{K_t^{(C,\xi)}(\omega)(C_t(\omega))^{1-\gamma} + n^{1-\gamma}}{(C_t^{m,n}(\omega))^{1-\gamma}} \leq \frac{K_t^{(C,\xi)}(\omega) n^{1-\gamma} + n^{1-\gamma}}{n^{1-\gamma}} = K_t^{(C,\xi)}(\omega) + 1.$$

In summary, for any rational number $t \in [0,T]$ and $\omega \in \Omega^*$, we have $\mathcal{D}_t^{m,n}(\omega) \leq K_t^{(C,\xi)}(\omega) + 1$. Since $\mathcal{D}^{m,n}$ and $K^{(C,\xi)}$ are càdlàg, we have

$$\mathcal{D}_t^{m,n} \leq K_t^{(C,\xi)} + 1, \tag{A.5}$$

for any $t \in [0,T]$ a.s. for any $m,n \in \mathbb{N}$. Since the function $x \to x \log x$ has a minimum value $-e^{-1}$, together with the bounds (A.5) and $U_t^{m,n} \geq U_t^{\mathrm{CRRA}}(C^{m,n}, \xi^{m,n})$, we have for any $m,n \in \mathbb{N}$ and $t \in [0,T]$,

$$e^{-1}\delta u_\gamma(C_t \wedge 1) \leq f(C_t^{m,n}, U_t^{m,n}) \leq -\delta u_\gamma(C_t \wedge 1)(K_t^{(C,\xi)} + 1)\log(K_t^{(C,\xi)} + 1),$$

a.s. From (2.10), we can see that $f(C_s^{m,n}, U_s^{m,n})$ is uniformly integrable $\mathrm{d}t \otimes \mathrm{d}\mathbb{P}$ on $[0,T] \times \Omega$. Furthermore, the comparison theorem for the Lipschitz approximating BSDEs implies that for any $m,n \in \mathbb{N}$, $U_t^{m+1,n} \leq U_t^{m,n} \leq U_t^{m,n+1}$ holds for all $t \in [0,T]$ a.s. Hence, there exists a stochastic process U^m such that

$$U_t^m = \lim_{n\to\infty} U_t^{m,n} = \sup_{n \in \mathbb{N}} U_t^{m,n},$$

for $t \in [0,T]$. This results in

$$U_t^m = \lim_{n\to\infty} U_t^{m,n} = \lim_{n\to\infty} \mathbb{E}_t\left[\xi^{m,n} + \int_t^T f(C_s^{m,n}, U_s^{m,n})\mathrm{d}s\right] = \mathbb{E}_t\left[\xi^m + \int_t^T f(C_s^m, U_s^m)\mathrm{d}s\right],$$

for any $t \in [0,T]$ and $m \in \mathbb{N}$, where $C_t^m = C_t \vee (1/m)$ and $\xi^m = u_\gamma(u_\gamma^{-1}(\xi) \vee (1/m))$. Since $U_t^m + \int_0^t f(C_s^m, U_s^m)\mathrm{d}s$ is a martingale and $\mathbb{F}$ satisfies the usual conditions, there exists a càdlàg version of U^m. Similarly, there exists a stochastic process U such that

$$U_t = \lim_{m\to\infty} U_t^m = \inf_{m\in\mathbb{N}} U_t^m,$$

for $t \in [0,T]$, and hence,

$$U_t = \lim_{m\to\infty} U_t^m = \lim_{m\to\infty} \mathbb{E}_t\left[\xi^m + \int_t^T f(C_s^m, U_s^m)\mathrm{d}s\right] = \mathbb{E}_t\left[\xi + \int_t^T f(C_s, U_s)\mathrm{d}s\right],$$

for any $t \in [0,T]$ a.s. From the same argument as above, there exists a càdlàg version of U. From the global bounds of $U^{m,n}$, it can be seen that U satisfies the inequality (2.12). Thus, U is uniformly integrable. Hence, U is a unit EIS Epstein–Zin SDU of (C, ξ). □

A.2 Proof of Proposition 2.5

Proof. In the proof, we consider the case $\gamma > 1$. The case $\gamma < 1$ can be proved similarly. Let

$$\alpha_t := \frac{\partial f(C_t, U_t^{\mathrm{sub}})}{\partial v} = -\delta\left[1 + \log\left(\frac{U_t^{\mathrm{sub}}}{u_\gamma(C_t)}\right)\right], \qquad t \in [0,T].$$

By the assumption in this proposition, there exists a constant $K > 0$ such that $\alpha_t \le -\delta(1 + \log K)$ holds for any $t \in [0,T]$ a.s. Let $\Delta U_t := U_t^{\mathrm{sub}} - U_t^{\mathrm{sup}}$ and

$$M_t^{\mathrm{sup}} := -\left(\Delta U_t + \int_0^t \left(f(C_s, U_s^{\mathrm{sub}}) - f(C_s, U_s^{\mathrm{sup}})\right)\mathrm{d}s - \Delta U_0\right), \qquad t \in [0,T].$$

Then, by the assumption, M^{sup} is a local supermartingale on $[0,T]$. Accordingly, M^{sup} admits the canonical decomposition such that

$$M_t^{\mathrm{sup}} = M_t^{\mathrm{loc.}} - A_t, \qquad t \in [0,T],$$

where $M^{\mathrm{loc.}}$ is a local martingale and A is a predictable increasing process with $M_0^{\mathrm{loc.}} = A_0 = 0$. For any $t \in [0,T]$ and stopping time τ on $[t,T]$, from the integration-by-parts formula, we have

$$\begin{aligned} e^{\int_0^t \alpha_s \mathrm{d}s}\Delta U_t = e^{\int_0^\tau \alpha_s \mathrm{d}s}\Delta U_\tau + \int_t^\tau e^{\int_0^s \alpha_{s'}\mathrm{d}s'}\left(f(C_s, U_s^{\mathrm{sub}}) - f(C_s, U_s^{\mathrm{sup}}) - \alpha_s \Delta U_s\right)\mathrm{d}s \\ + \int_t^\tau e^{\int_0^s \alpha_{s'}\mathrm{d}s'}\mathrm{d}M_s^{\mathrm{loc.}} - \int_t^\tau e^{\int_0^s \alpha_{s'}\mathrm{d}s'}\mathrm{d}A_s. \end{aligned}$$

Furthermore, since $f(c,v)$ is convex in v when $\gamma > 1$, it holds that

$$f(C_s, U_s^{\mathrm{sub}}) - f(C_s, U_s^{\mathrm{sup}}) - \alpha_s \Delta U_s \le 0, \qquad s \in [0,T].$$

Therefore, we have

$$\Delta U_t \le \mathbb{E}_t \left[e^{\int_t^{\tau_n} \alpha_s \mathrm{d}s} \Delta U_{\tau_n} \right],$$

for any $n \in \mathbb{N}$, where $\{\tau_n\}_{n\in\mathbb{N}}$ is a sequence of localizing stopping times such that

$$\tau_n := \inf\{s \ge t : |M_s^{\mathrm{loc.}} - M_t^{\mathrm{loc.}}| \ge n\} \wedge T.$$

Here, by the assumption, we have

$$e^{\int_t^{\tau_n} \alpha_s \mathrm{d}s} |\Delta U_{\tau_n}| \le e^{-\delta(1+\log K)(\tau_n - t)} \big(|U_{\tau_n}^{\mathrm{sub}}| + |U_{\tau_n}^{\mathrm{sup}}|\big) \le \big(1 \vee e^{-\delta(1+\log K)T}\big)\big(|U_{\tau_n}^{\mathrm{sub}}| + |U_{\tau_n}^{\mathrm{sup}}|\big),$$

and $\{|U_{\tau_n}^{\mathrm{sub}}| + |U_{\tau_n}^{\mathrm{sup}}|\}_{n\in\mathbb{N}}$ is uniformly integrable. Furthermore, $\tau_n \to T$ a.s. as $n \to \infty$. Hence, taking the limit as $n \to \infty$, the dominated convergence theorem yields

$$\Delta U_t \le \lim_{n\to\infty} \mathbb{E}_t \left[e^{\int_t^{\tau_n} \alpha_s \mathrm{d}s} \Delta U_{\tau_n} \right] = \mathbb{E}_t \left[e^{\int_t^{T} \alpha_s \mathrm{d}s} \Delta U_{T} \right] \le 0.$$

The last inequality is due to the assumption $U_T^{\mathrm{sub}} \le U_T^{\mathrm{sup}}$ a.s. Hence, we have obtained the desired result. □

A.3 Proof of Proposition 2.7

Proof. The homotheticity and monotonicity are immediate from Proposition 2.5. The concavity for $\gamma < 1$ is also immediate since $f(c,v)$ is jointly concave in $(c,v) \in (0,\infty) \times (0,\infty)$ when $\gamma < 1$. Thus, we shall prove concavity for $\gamma > 1$. The following proof is inspired by Proposition 2.4 in Xing (2017).

First consider a bounded case. Fix $(\widetilde{C}, \widetilde{\xi}), (\widehat{C}, \widehat{\xi}) \in \mathcal{C}_T^{\mathrm{B}}$ and a constant $p \in [0,1]$, arbitrarily. Then, $\big(p\widetilde{C} + (1-p)\widehat{C}, u_\gamma(pu_\gamma^{-1}(\widetilde{\xi}) + (1-p)u_\gamma^{-1}(\widehat{\xi}))\big) \in \mathcal{C}_T^{\mathrm{B}}$ holds. Let

$$\widetilde{Y}_t = \log \circ u_\gamma^{-1}\big(U_t(\widetilde{C}, \widetilde{\xi})\big), \quad \widehat{Y}_t = \log \circ u_\gamma^{-1}\big(U_t(\widehat{C}, \widehat{\xi})\big),$$

for $t \in [0,T]$. By construction, there exist $(\widetilde{Z}, \widetilde{\psi}), (\widehat{Z}, \widehat{\psi}) \in \mathbb{H}^2 \times \mathbb{J}^2$ such that $\widetilde{Y}$ and $\widehat{Y}$ satisfy

$$\widetilde{Y}_t = \widetilde{\xi}^Y + \int_t^T g(s, \widetilde{C}_s, \widetilde{Y}_s, \widetilde{Z}_s, \widetilde{\psi}_s)\mathrm{d}s - \int_t^T \widetilde{Z}_s^\top \mathrm{d}B_s - \int_t^T \int_J \widetilde{\psi}_s(x)\widetilde{\mu}(\mathrm{d}s, \mathrm{d}x),$$
$$\widehat{Y}_t = \widehat{\xi}^Y + \int_t^T g(s, \widehat{C}_s, \widehat{Y}_s, \widehat{Z}_s, \widehat{\psi}_s)\mathrm{d}s - \int_t^T \widehat{Z}_s^\top \mathrm{d}B_s - \int_t^T \int_J \widehat{\psi}_s(x)\widetilde{\mu}(\mathrm{d}s, \mathrm{d}x),$$

for $t \in [0,T]$, where $\widetilde{\xi}^Y = \log \circ u_\gamma^{-1}(\widetilde{\xi})$ and $\widehat{\xi}^Y = \log \circ u_\gamma^{-1}(\widehat{\xi})$. Let $\Delta_p Y_t := p\widetilde{Y}_t + (1-p)\widehat{Y}_t$. Similarly, we denote convex combinations of $(\widetilde{C}, \widehat{C})$, $(\widetilde{Z}, \widehat{Z})$, $(\widetilde{\psi}, \widehat{\psi})$, and $(\widetilde{\xi}^Y, \widehat{\xi}^Y)$ by $\Delta_p C$, $\Delta_p Z$, $\Delta_p \psi$, and

$\Delta_p\xi^Y$, respectively. Then, we have

$$\Delta_p Y_t = \Delta_p\xi^Y + \int_t^T \Big(g(s,\Delta_p C_s,\Delta_p Y_s,\Delta_p Z_s,\Delta_p\psi_s) + A_s^{(p)}\Big)\mathrm{d}s \\ - \int_t^T \Delta_p Z_s^\top \mathrm{d}B_s - \int_t^T\int_{\mathbb{R}^k}\Delta_p\psi_s(x)\widetilde{\mu}(\mathrm{d}s,\mathrm{d}x),$$

where

$$A_t^{(p)} = pg(t,\widetilde{C}_t,\widetilde{Y}_t,\widetilde{Z}_t,\widetilde{\psi}_t) + (1-p)g(t,\widehat{C}_t,\widehat{Y}_t,\widehat{Z}_t,\widehat{\psi}_t) - g(t,\Delta_p C_t,\Delta_p Y_t,\Delta_p Z_t,\Delta_p\psi_t).$$

Since $g(t,c,y,z,\phi)$ is concave in (c,y,z,ϕ), $A_t^{(p)} \le 0$ holds for any $t\in[0,T]$ a.s. Let

$$U_t^{\Delta_p} := u_\gamma(\exp\{\Delta_p Y_t\}) = \frac{1}{1-\gamma}\exp\Big\{(1-\gamma)\Delta_p Y_t\Big\}.$$

Then, $\Delta_p Y = p\widetilde{Y} + (1-p)\widehat{Y}$ is bounded since $(\widetilde{C},\widetilde{\xi}),(\widehat{C},\widehat{\xi})\in\mathcal{C}_T^{\mathrm{B}}$. Therefore, U^{Δ_p} is also bounded, that is, uniformly integrable. From the generalized Ito formula, we have

$$U_t^{\Delta_p} = \xi^{\Delta_p} + \int_t^T\Big(f(\Delta_p C_s, U_s^{\Delta_p}) + (1-\gamma)U_s^{\Delta_p}A_s^{(p)}\Big)\mathrm{d}s - (M_T^{\mathrm{loc.}} - M_t^{\mathrm{loc.}}),$$

for any $t\in[0,T]$, where $M^{\mathrm{loc.}}$ is a local martingale and $\xi^{\Delta_p} = u_\gamma(\exp\{\Delta_p\xi^Y\})$. By the uniform integrability of U^{Δ_p} and the non-positivity of $A_s^{(p)}\le 0$, similarly to the proof of Proposition 2.4, we have

$$U_t^{\Delta_p} \le \mathbb{E}_t\left[\xi^{\Delta_p} + \int_t^T f(\Delta_p C_s, U_s^{\Delta_p})\mathrm{d}s\right],$$

for any $t\in[0,T]$. Therefore, $U_t^{\Delta_p} + \int_0^t f(\Delta_p C_s, U_s^{\Delta_p})\mathrm{d}s$ is a local submartingale. Furthermore, since $(\widetilde{C},\widetilde{\xi})$ and $(\widehat{C},\widehat{\xi})$ are bounded, there exist two constants $\widetilde{K}$ and $\widehat{K}$ such that

$$\begin{aligned} U_t(\widetilde{C},\widetilde{\xi}) &\le \frac{1}{1-\gamma}\exp\left\{\mathbb{E}_t\left[\int_t^T \delta e^{-\delta(s-t)}\log\big(\widetilde{C}_s\big)^{1-\gamma}\mathrm{d}s + e^{-\delta(T-t)}\log((1-\gamma)\widetilde{\xi})\right]\right\} \\ &\le e^{(1-\gamma)\widetilde{K}}u_\gamma(\widetilde{C}_t), \\ U_t(\widehat{C},\widehat{\xi}) &\le \frac{1}{1-\gamma}\exp\left\{\mathbb{E}_t\left[\int_t^T \delta e^{-\delta(s-t)}\log\big(\widehat{C}_s\big)^{1-\gamma}\mathrm{d}s + e^{-\delta(T-t)}\log((1-\gamma)\widehat{\xi})\right]\right\} \\ &\le e^{(1-\gamma)\widehat{K}}u_\gamma(\widehat{C}_t), \end{aligned}$$

for any $t\in[0,T]$. These imply

$$\widetilde{Y}_t = \log\circ u_\gamma^{-1}\big(U_t(\widetilde{C},\widetilde{\xi})\big) \le \widetilde{K} + \log\widetilde{C}_t, \qquad \widehat{Y}_t = \log\circ u_\gamma^{-1}\big(U_t(\widehat{C},\widehat{\xi})\big) \le \widehat{K} + \log\widehat{C}_t.$$

Hence,

$$
\begin{aligned}
U_t^{\Delta_p} &= \frac{1}{1-\gamma}\exp\left\{(1-\gamma)\Big(p\widetilde{Y}_t + (1-p)\widehat{Y}_t\Big)\right\} \\
&\le \frac{1}{1-\gamma}\exp\left\{(1-\gamma)\Big[p(\widetilde{K}+\log\widetilde{C}_t) + (1-p)(\widehat{K}+\log\widehat{C}_t)\Big]\right\} \\
&= \frac{e^{(1-\gamma)(p\widetilde{K}+(1-p)\widehat{K})}}{1-\gamma}\exp\left\{(1-\gamma)\Big[p\log\widetilde{C}_t + (1-p)\log\widehat{C}_t\Big]\right\} \\
&\le \frac{e^{(1-\gamma)(p\widetilde{K}+(1-p)\widehat{K})}}{1-\gamma}\exp\Big\{(1-\gamma)\log\Delta_p C_t\Big\} = e^{(1-\gamma)(p\widetilde{K}+(1-p)\widehat{K})}u_\gamma(\Delta_p C_t).
\end{aligned}
$$

In summary, (1) U^{Δ_p} is uniformly integrable, (2) $U_t^{\Delta_p} + \int_0^t f(\Delta_p C_s, U_s^{\Delta_p})\mathrm{d}s$ is a local submartingale, and (3) $U_t^{\Delta_p} \le e^{(1-\gamma)(p\widetilde{K}+(1-p)\widehat{K})}u_\gamma(\Delta_p C_t)$ holds for any $t\in[0,T]$ a.s.

Let $\xi^{(p)} = u_\gamma(pu_\gamma^{-1}(\widetilde{\xi}) + (1-p)u_\gamma^{-1}(\widehat{\xi}))$ and $U^{(p)} = U(\Delta_p C, \xi^{(p)})$. Then,

$$
U_t^{(p)} = \mathbb{E}_t\left[\xi^{(p)} + \int_t^T f(\Delta_p C_s, U_s^{(p)})\mathrm{d}s\right],
$$

for any $t\in[0,T]$. Thus, $U_t^{(p)} + \int_0^t f(\Delta_p C_s, U_s^{(p)})\mathrm{d}s$ is a local martingale. Furthermore, $U^{(p)}$ is bounded, and hence uniformly integrable. Moreover, since $\exp\{x\}$ is convex and $u_\gamma(x)$ is increasing, we have $\xi^{(p)} \ge \xi^{\Delta_p}$ a.s. Hence, Proposition 2.5 implies $U_t^{(p)} \ge U_t^{\Delta_p}$ for any $t\in[0,T]$. Finally, we have

$$
\begin{aligned}
&U_t\Big(p\widetilde{C} + (1-p)\widehat{C}, u_\gamma\big(pu_\gamma^{-1}(\widetilde{\xi}) + (1-p)u_\gamma^{-1}(\widehat{\xi})\big)\Big) = U_t^{(p)} \\
&\quad\ge U_t^{\Delta_p} = \frac{1}{1-\gamma}\exp\left\{p\log\Big((1-\gamma)U_t(\widetilde{C},\widetilde{\xi})\Big) + (1-p)\log\Big((1-\gamma)U_t(\widehat{C},\widehat{\xi})\Big)\right\} \\
&\quad\ge \frac{1}{1-\gamma}\exp\left\{\log\Big((1-\gamma)\big(pU_t(\widetilde{C},\widetilde{\xi}) + (1-p)U_t(\widehat{C},\widehat{\xi})\big)\Big)\right\} \\
&\quad= pU_t(\widetilde{C},\widetilde{\xi}) + (1-p)U_t(\widehat{C},\widehat{\xi}),
\end{aligned}
$$

where the last inequality is due to the decreasing property of $x\to e^x/(1-\gamma)$ and the concavity of $x\to\log x$. Thus, in the bounded case, the desired inequality is obtained.

In the unbounded case, fix $(\widetilde{C},\widetilde{\xi}), (\widehat{C},\widehat{\xi}) \in \mathcal{C}_T^{\mathrm{E}}$ and $p\in[0,1]$, arbitrarily. For any positive integers m, n, let

$$
\begin{aligned}
\widetilde{C}_t^{m,n} &:= (\widetilde{C}_t \vee 1/m)\wedge n, \quad \widehat{C}_t^{m,n} := (\widehat{C}_t \vee 1/m)\wedge n, \quad \Delta_p^{m,n}C_t := p\widetilde{C}_t^{m,n} + (1-p)\widehat{C}_t^{m,n} \quad t\in[0,T], \\
\widetilde{\xi}^{m,n} &:= u_\gamma\Big(\big(u_\gamma^{-1}(\widetilde{\xi})\vee 1/m\big)\wedge n\Big), \quad \widehat{\xi}^{m,n} := u_\gamma\Big(\big(u_\gamma^{-1}(\widehat{\xi})\vee 1/m\big)\wedge n\Big), \\
\xi^{(p),m,n} &:= u_\gamma\big(pu_\gamma^{-1}(\widetilde{\xi}^{m,n}) + (1-p)u_\gamma^{-1}(\widehat{\xi}^{m,n})\big).
\end{aligned}
$$

Then, from the same argument as in the bounded case, we have

$$
U_t(\Delta_p^{m,n}C, \xi^{(p),m,n}) \ge pU_t(\widetilde{C}^{m,n},\widetilde{\xi}^{m,n}) + (1-p)U_t(\widehat{C}^{m,n},\widehat{\xi}^{m,n}), \tag{A.6}
$$

for all $t \in [0, T]$. By definition, we have

$$(1-\gamma)\xi^{(p),m,n} = \left(pu_\gamma^{-1}(\widetilde{\xi}^{m,n}) + (1-p)u_\gamma^{-1}(\widehat{\xi}^{m,n})\right)^{1-\gamma} \leq p(1-\gamma)\widetilde{\xi}^{m,n} + (1-p)(1-\gamma)\widehat{\xi}^{m,n} \leq m^{-(1-\gamma)}, \tag{A.7}$$

and

$$\begin{aligned}
(1-\gamma)\xi^{(p),m,n} &= \left(p\Big((u_\gamma^{-1}(\widetilde{\xi}) \vee 1/m) \wedge n\Big) + (1-p)\Big((u_\gamma^{-1}(\widehat{\xi}) \vee 1/m) \wedge n\Big)\right)^{1-\gamma} \\
&\leq \Big(p\big(u_\gamma^{-1}(\widetilde{\xi}) \wedge n\big) + (1-p)\big(u_\gamma^{-1}(\widehat{\xi}) \wedge n\big)\Big)^{1-\gamma} \\
&= \Big(pu_\gamma^{-1}\big(\widetilde{\xi} \wedge u_\gamma(n)\big) + (1-p)u_\gamma^{-1}\big(\widehat{\xi} \wedge u_\gamma(n)\big)\Big)^{1-\gamma} \\
&\leq p\Big(u_\gamma^{-1}\big(\widetilde{\xi} \wedge u_\gamma(n)\big)\Big)^{1-\gamma} + (1-p)\Big(u_\gamma^{-1}\big(\widehat{\xi} \wedge u_\gamma(n)\big)\Big)^{1-\gamma} \\
&= p\Big(((1-\gamma)\widetilde{\xi}) \vee n^{1-\gamma}\Big) + (1-p)\Big(((1-\gamma)\widehat{\xi}) \vee n^{1-\gamma}\Big) \\
&\leq (1-\gamma)(p\widetilde{\xi} + (1-p)\widehat{\xi}) + n^{1-\gamma}.
\end{aligned} \tag{A.8}$$

From the inequalities (A.7) and (A.8) and the assumption that $(\Delta_p C, p\widetilde{\xi} + (1-p)\widehat{\xi})$ satisfies (2.9), we can apply the same argument as in Step 4 of the proof of Proposition 2.4. Thus, taking the limit of (A.6) as $m \to \infty$ and $n \to \infty$, we obtain the desired result. □

A.4 Proof of Proposition 2.9

Proof. Fix $C \in \mathcal{C}_T$, and $\theta > 0$, arbitrarily. Suppose that C is bounded above and away from zero. Let $f_\theta(c, v)$ be a unit EIS Epstein–Zin generator with RRA $1 + 1/\theta$. The inequality $u_{1+1/\theta}(\exp\{Y^{\text{rob.},\theta}(C)\}) \geq U^{\text{EZ},1+1/\theta}(C)$ is relatively easy to show. Meanwhile, the opposite inequality $u_{1+1/\theta}(\exp\{Y^{\text{rob.},\theta}(C)\}) \leq U^{\text{EZ},1+1/\theta}(C)$ is more involved.

1. The easy inequality. Fix $(\widetilde{Z}, \widetilde{\psi}) \in \mathbb{M}^\infty$, arbitrarily. Consider a BSDE with respect to $(Y^{(\widetilde{Z},\widetilde{\psi})}(C), \widehat{Z}, \widehat{\psi})$ on $(\Omega, \mathcal{F}, \mathbb{Q}^{(\widetilde{Z},\widetilde{\psi})})$ such that

$$\begin{aligned}
Y_t^{(\widetilde{Z},\widetilde{\psi})}(C) = \log C_T &+ \int_t^T \left\{\delta\Big(\log C_s - Y_s^{(\widetilde{Z},\widetilde{\psi})}(C)\Big) + \theta R_s(\widetilde{Z}_s, \widetilde{\psi}_s)\right\} \mathrm{d}s \\
&- \int_t^T \widehat{Z}_s^\top \mathrm{d}B_s^{(\widetilde{Z},\widetilde{\psi})} - \int_t^T \int_J \widehat{\psi}_s(x)\widetilde{\mu}^{(\widetilde{Z},\widetilde{\psi})}(\mathrm{d}s, \mathrm{d}x).
\end{aligned}$$

We can see that this Lipschitz BSDE has a solution in $\mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{J}^2$ under $\mathbb{Q}^{(\widetilde{Z},\widetilde{\psi})}$. Since C is bounded and $(\widetilde{Z}, \widetilde{\psi}) \in \mathbb{M}^\infty$, we can easily see that $Y^{(\widetilde{Z},\widetilde{\psi})}(C) \in \mathbb{S}^\infty$ under $\mathbb{Q}^{(\widetilde{Z},\widetilde{\psi})}$. It can be also seen that $\|\widehat{\psi}\|_{\mathbb{J}^\infty} \leq 2\|Y^{(\widetilde{Z},\widetilde{\psi})}(C)\|_{\mathbb{S}^\infty} < \infty$, so $\widehat{\psi} \in \mathbb{J}^\infty$ under $\mathbb{Q}^{(\widetilde{Z},\widetilde{\psi})}$. Since $\mathbb{Q}^{(\widetilde{Z},\widetilde{\psi})}$ is equivalent to $\mathbb{P}$, we have $(Y^{(\widetilde{Z},\widetilde{\psi})}(C), \widehat{\psi}) \in \mathbb{S}^\infty \times \mathbb{J}^\infty$ and $\int_0^T \|\widehat{Z}_s\|^2 \mathrm{d}s < \infty$ a.s. under $\mathbb{P}$. Furthermore, it can be transformed to the representation on $(\Omega, \mathcal{F}, \mathbb{P})$ as follows:

$$\begin{aligned} Y_t^{(\widetilde{Z},\widetilde{\psi})}(C) = \log C_T + \int_t^T \Big\{ \delta\Big(\log C_s - Y_s^{(\widetilde{Z},\widetilde{\psi})}(C) \Big) + \widehat{R}_s(\widetilde{Z}_s, \widetilde{\psi}_s, \widehat{Z}_s, \widehat{\psi}_s) \Big\} \mathrm{d}s \\ - \int_t^T \widehat{Z}_s^\top \mathrm{d}B_s - \int_t^T \int_J \widehat{\psi}_s(x) \widetilde{\mu}(\mathrm{d}s, \mathrm{d}x), \end{aligned} \tag{A.9}$$

where

$$\begin{aligned} \widehat{R}_s(\widetilde{Z}_s, \widetilde{\psi}_s, \widehat{Z}_s, \widehat{\psi}_s) &:= \theta R_s(\widetilde{Z}_s, \widetilde{\psi}_s) + \widehat{Z}_s^\top \widetilde{Z}_s + \int_J \widehat{\psi}_s(x) \widetilde{\psi}_s(x) \lambda_s(x) \zeta(\mathrm{d}x) \\ &= \frac{\theta}{2} \|\widetilde{Z}_s\|^2 + \widehat{Z}_s^\top \widetilde{Z}_s + \int_J r_3\big(\widetilde{\psi}(x), \widehat{\psi}(x)\big) \lambda_s(x) \zeta(\mathrm{d}x), \\ r_3(y,x) &:= \theta\big((1+y)\log(1+y) - y\big) + xy. \end{aligned}$$

Since $y \to r_3(y,x)$ is convex on $(-1, \infty)$, by the Fenchel–Young inequality, we have $r_3(y,x) \geq j_{-1/\theta}(x)$, for any $x \in \mathbb{R}$ and $y > -1$, where the equality holds when $y = e^{-x/\theta} - 1$. Similarly, we have $\frac{\theta}{2}\|\widetilde{z}\|^2 + \widehat{z}^\top \widetilde{z} \geq \frac{-1/\theta}{2}\|\widehat{z}\|^2$, for any $\widetilde{z}, \widehat{z} \in \mathbb{R}^d$, where the equality holds when $\widetilde{z} = -\widehat{z}/\theta$. Accordingly, we have

$$\widehat{R}_s(\widetilde{Z}_s, \widetilde{\psi}_s, \widehat{Z}_s, \widehat{\psi}_s) \geq \underline{R}_s(\widehat{Z}_s, \widehat{\psi}_s) := \frac{-1/\theta}{2} \|\widehat{Z}_s\|^2 + \int_J j_{-1/\theta}\big(\widehat{\psi}_s(x)\big) \lambda_s(x) \zeta(\mathrm{d}x).$$

Let $U_t^{(\widetilde{Z},\widetilde{\psi}),\theta}(C) := u_{1+1/\theta}(\exp\{Y_t^{(\widetilde{Z},\widetilde{\psi})}(C)\})$. Then, from the generalized Ito formula, we have

$$\begin{aligned} U_t^{(\widetilde{Z},\widetilde{\psi}),\theta}(C) &= u_{1+1/\theta}(C_T) \\ &\quad + \int_t^T \bigg(f_\theta\big(C_s, U_s^{(\widetilde{Z},\widetilde{\psi}),\theta}(C)\big) + e^{-Y_s^{(\widetilde{Z},\widetilde{\psi})}(C)/\theta} \Big(\widehat{R}_s(\widetilde{Z}_s, \widetilde{\psi}_s, \widehat{Z}_s, \widehat{\psi}_s) - \underline{R}_s(\widehat{Z}_s, \widehat{\psi}_s) \Big) \bigg) \mathrm{d}s \\ &\quad - \int_t^T e^{-Y_s^{(\widetilde{Z},\widetilde{\psi})}(C)/\theta} \widehat{Z}_s^\top \mathrm{d}B_s - \int_t^T \int_J u_{1+1/\theta}(\exp\{Y_{s-}^{(\widetilde{Z},\widetilde{\psi})}(C)\}) \Big(e^{-\widehat{\psi}_s(x)/\theta} - 1 \Big) \widetilde{\mu}(\mathrm{d}s, \mathrm{d}x). \end{aligned}$$

Note that $(Y^{(\widetilde{Z},\widetilde{\psi})}(C), \widehat{\psi}) \in \mathbb{S}^\infty \times \mathbb{J}^\infty$ and $\int_0^T \|\widehat{Z}_s\|^2 \mathrm{d}s < \infty$ a.s. under $\mathbb{P}$. It follows that $U_t^{(\widetilde{Z},\widetilde{\psi}),\theta}(C) + \int_0^t f_\theta(C_s, U_s^{(\widetilde{Z},\widetilde{\psi}),\theta}(C)) \mathrm{d}s$ is a local supermartingale under $\mathbb{P}$. Meanwhile, by definition, we can see that $U_t^{\mathrm{EZ},1+1/\theta}(C) + \int_0^t f_\theta(C_s, U_s^{\mathrm{EZ},1+1/\theta}(C)) \mathrm{d}s$ is a local martingale under $\mathbb{P}$ with $U_T^{\mathrm{EZ},1+1/\theta}(C) = U_T^{(\widetilde{Z},\widetilde{\psi}),\theta}(C) = u_{1+1/\theta}(C_T)$. Moreover, there exists a constant $K > 0$ such that $U_t^{\mathrm{EZ},1+1/\theta}(C) \leq K u_{1+1/\theta}(C_t)$ holds for any $t \in [0,T]$ since C is bounded. Therefore, from Proposition 2.5, $U_t^{\mathrm{EZ},1+1/\theta}(C) \leq U_t^{(\widetilde{Z},\widetilde{\psi}),\theta}(C)$ holds for any $t \in [0,T]$ a.s. on the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Thus, we have $u_{1+1/\theta}(\exp\{Y_t^{\mathrm{rob.},\theta}(C)\}) \geq U_t^{\mathrm{EZ},1+1/\theta}(C)$ for any $t \in [0,T]$ a.s. on $(\Omega, \mathcal{F}, \mathbb{P})$.

2. The hard inequality: Approximations. Next, we shall prove $u_{1+1/\theta}(\exp\{Y_t^{\mathrm{rob.},\theta}(C)\}) \leq U_t^{\mathrm{EZ},1+1/\theta}(C)$. To obtain a globally Lipschitz driver while preserving monotone convergence to the

original quadratic–exponential driver, we define the following for any positive integer n:

$$R_t^n(z,\phi) := \frac{\theta}{2}\|\widetilde{\mathcal{Z}}_n(z)\|^2 + z^\top \widetilde{\mathcal{Z}}_n(z) + \int_J r_3\Big(\widetilde{\varphi}_n\big(\phi(x)\big), \phi(x)\Big)\lambda_t(x)\zeta(\mathrm{d}x), \qquad (z,\phi)\in\mathbb{R}^d\times\mathbb{L},$$
$$\widetilde{\mathcal{Z}}_n(z) := -\frac{\rho_n(\|z\|^2)}{\theta}z,$$

where $\rho_n(x)$ is defined in the proof of Proposition 2.4 and the function $\widetilde{\varphi}_n$ from $\mathbb{R}$ to $\mathbb{R}$ is defined as

$$\widetilde{\varphi}_n(x) := \begin{cases} e^{-x/\theta}-1, & x \ge -n, \\ e^{-p_n(x)/\theta}-1, & x\in(-(n+1),-n), \\ e^{(n+1/2)/\theta}-1, & x\le -(n+1), \end{cases} \qquad p_n(x) := x + \frac{1}{2}(x+n)^2.$$

Then, $\widetilde{\varphi}_n$ is bounded, non-increasing, Lipschitz continuous, and continuously differentiable for any $n\in\mathbb{N}$. Furthermore, $\widetilde{\varphi}_n(x)\le\widetilde{\varphi}_{n+1}(x)\le e^{-x/\theta}-1$ holds for any $x\in\mathbb{R}$ and $n\in\mathbb{N}$, and $\lim_{n\to\infty}\widetilde{\varphi}_n(x) = e^{-x/\theta}-1$ for any $x\in\mathbb{R}$. Therefore, $R_t^n(z,\phi)$ is Lipschitz continuous in (z,ϕ), and for each $n\in\mathbb{N}$, the Lipschitz coefficient is uniformly determined in (t,ω). Here, consider a BSDE with respect to $(\widetilde{Y}^n(C),\widetilde{Z}^n,\widetilde{\psi}^n)$ such that

$$\begin{aligned}\widetilde{Y}_t^n(C) = \log C_T &+ \int_t^T \Big\{\delta\Big(\log C_s - \widetilde{Y}_s^n(C)\Big) + R_s^n(\widetilde{Z}_s^n,\widetilde{\psi}_s^n)\Big\}\,\mathrm{d}s \\ &- \int_t^T (\widetilde{Z}_s^n)^\top \mathrm{d}B_s - \int_t^T\int_J \widetilde{\psi}_s^n(x)\widetilde{\mu}(\mathrm{d}t,\mathrm{d}x).\end{aligned} \tag{A.10}$$

Since the BSDE (A.10) is Lipschitz, there exists a solution $(\widetilde{Y}^n(C),\widetilde{Z}^n,\widetilde{\psi}^n)\in\mathbb{S}^2\times\mathbb{H}^2\times\mathbb{J}^2$. Since C is bounded, the classical a priori estimate for Lipschitz BSDEs implies $\|\widetilde{Y}^n(C)\|_{\mathbb{S}^\infty}<\infty$. As a direct consequence, we also have $\|\widetilde{\psi}^n\|_{\mathbb{J}^\infty}\le 2\|\widetilde{Y}^n(C)\|_{\mathbb{S}^\infty}<\infty$. Furthermore, we have

$$\widehat{R}_s\Big(\widetilde{\mathcal{Z}}_n(\widetilde{Z}_s^n),\widetilde{\varphi}_n(\widetilde{\psi}_s^n),\widetilde{Z}_s^n,\widetilde{\psi}_s^n\Big) = R_s^n(\widetilde{Z}_s^n,\widetilde{\psi}_s^n),$$

and for any $(\widehat{Z},\widehat{\psi})\in\mathbb{H}^2\times\mathbb{J}^2$,

$$\begin{aligned}&\widehat{R}_s\Big(\widetilde{\mathcal{Z}}_n(\widetilde{Z}_s^n),\widetilde{\varphi}_n(\widetilde{\psi}_s^n),\widehat{Z}_s,\widehat{\psi}_s\Big) - R_s^n(\widetilde{Z}_s^n,\widetilde{\psi}_s^n) \\ &= \Big(\widetilde{\mathcal{Z}}_n(\widetilde{Z}_s^n)\Big)^\top\Big(\widehat{Z}_s-\widetilde{Z}_s^n\Big) + \int_J\widetilde{\varphi}_n(\widetilde{\psi}_s^n)\Big(\widehat{\psi}_s(x)-\widetilde{\psi}_s^n(x)\Big)\lambda_s(x)\zeta(\mathrm{d}x).\end{aligned}$$

Moreover, $\widetilde{\mathcal{Z}}_n(\widetilde{Z}^n)$ and $\widetilde{\varphi}_n(\widetilde{\psi}^n)$ are bounded, and it holds that under $\mathbb{P}$,

$$\widetilde{\varphi}_n\big(\widetilde{\psi}_t^n(x)\big)\ge\exp\{-\|\widetilde{\psi}^n\|_{\mathbb{J}^\infty}/\theta\}-1>-1, \qquad (t,x)\in[0,T]\times J.$$

Thus, $(\widetilde{\mathcal{Z}}_n(\widetilde{Z}^n),\widetilde{\varphi}_n(\widetilde{\psi}^n))\in\mathbb{M}^\infty$. Hence, there exists a solution to the BSDE (A.9) with $(\widetilde{Z},\widetilde{\psi}) = (\widetilde{\mathcal{Z}}_n(\widetilde{Z}^n),\widetilde{\varphi}_n(\widetilde{\psi}^n))$, denoted by $(Y^{(\widetilde{\mathcal{Z}}_n(\widetilde{Z}^n),\widetilde{\varphi}_n(\widetilde{\psi}^n))}(C),\widehat{Z}^n,\widehat{\psi}^n)\in\mathbb{S}^2\times\mathbb{H}^2\times\mathbb{J}^2$. Then, the comparison theorem for Lipschitz BSDEs under $\mathbb{P}$ implies $\widetilde{Y}_t^n(C) = Y_t^{(\widetilde{\mathcal{Z}}_n(\widetilde{Z}^n),\widetilde{\varphi}_n(\widetilde{\psi}^n))}(C)$ for any $t\in[0,T]$ a.s.[3]

[3]In this case, we may apply a comparison theorem for Lipschitz BSDEs that does not require the A_Γ condition, because the generator $(t,y,z,\psi)\to\delta(\log C_t - y)+\widehat{R}_t(\widetilde{\mathcal{Z}}_n(\widetilde{Z}_t^n),\widetilde{\varphi}_n(\widetilde{\psi}_t^n),z,\psi)$ satisfies the assumptions of Theorem

3. The hard inequality: Monotonicity. We shall prove $0 \geq R_t^n(z,\phi) \geq R_t^{n+1}(x,\phi)$ for any $(t,z,\phi,n) \in [0,T] \times \mathbb{R}^d \times \mathbb{L} \times \mathbb{N}$, a.s. $R_t^n(z,\phi)$ can be divided as follows.

$$R_t^n(z,\phi) = r_1^n(z) + r_{2,t}^n(\phi), \qquad r_1^n(z) := -\frac{1}{2\theta}\|z\|^2\rho_n(\|z\|^2)\Big(2-\rho_n(\|z\|^2)\Big),$$
$$r_{2,t}^n(\phi) := \int_J r_3\Big(\widetilde{\varphi}_n\big(\phi(x)\big), \phi(x)\Big)\lambda_t(x)\zeta(\mathrm{d}x), \qquad (z,\phi) \in \mathbb{R}^d \times \mathbb{L}.$$

It is clear that $0 \geq r_1^n(z) \geq r_1^{n+1}(z)$ for any $z \in \mathbb{R}^d$ and $n \in \mathbb{N}$. To examine $r_{2,t}^n$, for any $y \leq e^{-x/\theta}-1$, we have

$$\frac{\partial r_3(y,x)}{\partial y} = \theta \log\big(1+y\big) + x \leq \theta \log e^{-x/\theta} + x = 0.$$

Since $\widetilde{\varphi}_n(x) \leq \widetilde{\varphi}_{n+1}(x) \leq e^{-x/\theta}-1$ for any $x \in \mathbb{R}$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} r_3\Big(\widetilde{\varphi}_n\big(\phi(x)\big), \phi(x)\Big) \geq r_3\Big(\widetilde{\varphi}_{n+1}\big(\phi(x)\big), \phi(x)\Big) \geq r_3\big(e^{-\phi(x)/\theta}-1, \phi(x)\big) = \frac{e^{-\phi(x)/\theta}-1}{-1/\theta} - \phi(x) \\ = j_{-1/\theta}\big(\phi(x)\big), \end{aligned}$$

for any $x \in \mathbb{R}$, $n \in \mathbb{N}$, and $\phi \in \mathbb{L}$. This implies that $r_{2,t}^n(\phi) \geq r_{2,t}^{n+1}(\phi)$. We next show the non-positivity of $r_{2,t}^n$. If $x \geq -n$, then $\widetilde{\varphi}_n(x) = e^{-x/\theta}-1$. Hence, for any $\phi \in \mathbb{L}$,

$$r_3\Big(\widetilde{\varphi}_n\big(\phi(x)\big), \phi(x)\Big) = \frac{e^{-\phi(x)/\theta}-1}{-1/\theta} - \phi(x) = j_{-1/\theta}\big(\phi(x)\big) \leq 0, \qquad \phi(x) \geq -n.$$

Meanwhile, we have

$$\frac{\partial r_3(\widetilde{\varphi}_n(x), x)}{\partial x} = \Big(\theta \log\big(\widetilde{\varphi}_n(x)+1\big) + x\Big)\widetilde{\varphi}_n'(x) + \widetilde{\varphi}_n(x), \qquad x \in \mathbb{R}.$$

We can see that $\theta \log(\widetilde{\varphi}_n(x)+1) + x \leq \theta \log e^{-x/\theta} + x = 0$ and $\widetilde{\varphi}_n'(x) \leq 0$ hold for any $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Moreover, $\widetilde{\varphi}_n(x) > 0$ also holds if $x < 0$. Thus, $\partial r_3(\widetilde{\varphi}_n(x), x)/\partial x > 0$ holds for any $x < -n$. This implies that for any $\phi \in \mathbb{L}$,

$$r_3\Big(\widetilde{\varphi}_n\big(\phi(x)\big), \phi(x)\Big) < r_3\big(\widetilde{\varphi}_n(-n), -n\big) = \frac{e^{n/\theta}-1}{-1/\theta} + n = j_{-1/\theta}(-n) < 0, \qquad \phi(x) < -n$$

Therefore, $0 \geq r_{2,t}^n(\phi)$ holds for any $(t,\phi,n) \in [0,T] \times \mathbb{L} \times \mathbb{N}$, a.s. Thus, we obtain $0 \geq R_t^n(z,\phi) \geq R_t^{n+1}(z,\phi)$ for any $(t,z,\phi,n) \in [0,T] \times \mathbb{R}^d \times \mathbb{L} \times \mathbb{N}$, a.s. This monotonicity yields $R_t^n(z,\phi) \downarrow \underline{R}_t(z,\phi)$ as $n \to \infty$ for any $(t,z,\phi) \in [0,T] \times \mathbb{R}^d \times \mathbb{L}$ a.s. from the monotone convergence theorem. Furthermore, it holds that

$$-\delta|\log C_t| - \delta|y| + \underline{R}_t(z,\phi) \leq \delta(\log C_t - y) + R_t^n(z,\phi) \leq \delta|\log C_t| + \delta|y|, \tag{A.11}$$

for any $(t,y,z,\phi) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}$.

4. The hard inequality: A_Γ condition. We shall examine the A_Γ condition. From the funda-

3.2.2 in Delong (2013).

mental theorem of calculus, we have

$$R_t^n(z,\phi)-R_t^n(z,\widetilde{\phi})=\int_J\bigg(\int_0^1\frac{\mathrm{d}r_3(\widetilde{\varphi}_n(w),w)}{\mathrm{d}w}\bigg|_{w=k\phi(x)+(1-k)\widetilde{\phi}(x)}\mathrm{d}k\bigg)\big(\phi(x)-\widetilde{\phi}(x)\big)\lambda_t(x)\zeta(\mathrm{d}x),$$

for any $(t,z,\phi,\widetilde{\phi},n)\in[0,T]\times\mathbb{R}^d\times\mathbb{L}\times\mathbb{L}\times\mathbb{N}$. For any $|w|\le K$, we have

$$-1<e^{-K/\theta}-1\le\frac{\mathrm{d}r_3(\widetilde{\varphi}_n(w),w)}{\mathrm{d}w}=\Big(\theta\log\big(\widetilde{\varphi}_n(w)+1\big)+w\Big)\widetilde{\varphi}_n'(w)+\widetilde{\varphi}_n(w).$$

If $w\in[0,K]$, we have $\mathrm{d}r_3(\widetilde{\varphi}_n(w),w)/\mathrm{d}w\le e^{K/\theta}-1$. If $w\in[-K,0)$, then it can be seen that $0\ge\theta\log(\widetilde{\varphi}_n(w)+1)+w\ge-2K$ and $0\ge\widetilde{\varphi}_n'(w)\ge-K_\theta e^{n/\theta}$ hold where $K_\theta:=e^{1/(2\theta)}/\theta>0$. Furthermore, $\theta\log(\widetilde{\varphi}_n(w)+1)+w=0$ if $w>-n$ and $\widetilde{\varphi}_n'(w)=0$ if $w<-(n+1)$. Thus, for any $|w|\le K$,

$$\begin{aligned}\frac{\mathrm{d}r_3(\widetilde{\varphi}_n(w),w)}{\mathrm{d}w}&\le 2KK_\theta e^{n/\theta}\mathbb{1}_{\{-(n+1)\le w\le -n\}}+e^{K/\theta}-1\\&\le 2KK_\theta e^{K/\theta}+e^{K/\theta}-1=(2KK_\theta+1)e^{K/\theta}-1<\infty.\end{aligned}$$

Let

$$\Gamma_t^{n,\phi,\widetilde{\phi}}(x)=\int_0^1\frac{\mathrm{d}r_3(\widetilde{\varphi}_n(w),w)}{\mathrm{d}w}\bigg|_{w=k\phi(x)+(1-k)\widetilde{\phi}(x)}\mathrm{d}k.$$

Then, for any $\phi,\widetilde{\phi}\in\mathbb{L}$ with $|\phi(x)|\vee|\widetilde{\phi}(x)|\le K$, $\nu(\{t\},\cdot)$-a.e., we have

$$-1<e^{-K/\theta}-1\le\Gamma_t^{n,\phi,\widetilde{\phi}}(x)\le(2K_\theta K+1)e^{K/\theta}-1<\infty,\qquad\nu(\{t\},\cdot)\text{-a.e.},$$

and

$$R_t^n(z,\phi)-R_t^n(z,\widetilde{\phi})\le\int_J\Gamma_t^{n,\phi,\widetilde{\phi}}(x)\Big(\phi(x)-\widetilde{\phi}(x)\Big)\lambda_t(x)\zeta(\mathrm{d}x),$$

for any $(t,z,n)\in[0,T]\times\mathbb{R}^d\times\mathbb{N}$. Moreover, by the definition of $\widetilde{\varphi}_n$, $r_3(e^{-\phi/\theta}-1,\phi)=r_3(\widetilde{\varphi}_n(\phi),\phi)$ holds for any $\phi\in\mathbb{L}$ and $n\in\mathbb{N}$ with $|\phi|\le n$. Accordingly, $\Gamma_t^{n,\phi,\widetilde{\phi}}=\Gamma_t^{m,\phi,\widetilde{\phi}}$ holds $\nu(\{t\},\cdot)$-a.e. for any $(t,\phi,\widetilde{\phi})\in[0,T]\times\mathbb{L}\times\mathbb{L}$ and $n,m\in\mathbb{N}$ with $n\le m$ and $|\phi(x)|\vee|\widetilde{\phi}(x)|\le n$, $\nu(\{t\},\cdot)$-a.e. Thus, the A_Γ condition is satisfied, uniformly in sufficiently large n.

From the monotonicity of R^n in n and A_Γ condition, the comparison theorem for Lipschitz BSDEs yields $\widetilde{Y}_t^n(C)\ge\widetilde{Y}_t^{n+1}(C)$ for any $t\in[0,T]$ and $n\in\mathbb{N}$ a.s. under $\mathbb{P}$. Meanwhile, it can be seen that $\|\log\circ u_{1+1/\theta}^{-1}(U_t^{\mathrm{EZ},1+1/\theta})(C)\|_{\mathbb{S}^\infty}<\infty$ holds under $\mathbb{P}$ because C is bounded. Together with Step 1, we have

$$-\infty<-\|\log\circ u_{1+1/\theta}^{-1}(U_t^{\mathrm{EZ},1+1/\theta})(C)\|_{\mathbb{S}^\infty}\le\inf_{(\widetilde{Z},\widetilde{\psi})\in\mathbb{M}^\infty}Y_t^{(\widetilde{Z},\widetilde{\psi})}(C)\le\widetilde{Y}_t^n(C)\le\|\widetilde{Y}_t^1(C)\|_{\mathbb{S}^\infty}<\infty,\tag{A.12}$$

for any $t\in[0,T]$ and $n\in\mathbb{N}$, a.s. under $\mathbb{P}$. Thus, $\{\widetilde{Y}^n(C)\}_{n\in\mathbb{N}}$ is bounded uniformly in $n\in\mathbb{N}$.

5. The hard inequality: Convergence. We shall show

$$\lim_{n\to\infty}\widetilde{Y}_t^n(C)=\log\circ u_{1+1/\theta}^{-1}\big(U_t^{\mathrm{EZ},1+1/\theta}(C)\big),$$

a.s. on $(\Omega, \mathcal{F}, \mathbb{P})$. It is clear that $R^n_t(z,\phi) \to \underline{R}_t(z,\phi)$ as $n \to \infty$ for any $(z,\phi) \in \mathbb{R}^d \times \mathbb{L}$. By Step 4, $\{\widetilde{Y}^n(C)\}_{n\in\mathbb{N}}$ is uniformly bounded in $\mathbb{S}^\infty$ as in (A.12) and the uniform A_Γ condition holds for sufficiently large n. Moreover, $\{R^n\}_{n\in\mathbb{N}}$ satisfies the uniform quadratic–exponential growth bound (A.11). Furthermore, both of $\{\widetilde{Y}^n(C)\}_{n\in\mathbb{N}}$ and $\{R^n\}_{n\in\mathbb{N}}$ are non-increasing. Thus, all conditions in Proposition 4.1 in Fujii and Takahashi (2018) are satisfied. Hence, there exists a solution to the following BSDE:

$$\widetilde{Y}_t(C) = \log C_T + \int_t^T \Big(\delta\big(\log C_s - \widetilde{Y}_s(C)\big) + \underline{R}_s(\widetilde{Z}_s, \widetilde{\psi}_s)\Big)\mathrm{d}s - \int_t^T \widetilde{Z}_s^\top \mathrm{d}B_s - \int_t^T\int_J \widetilde{\psi}_s(x)\widetilde{\mu}(\mathrm{d}s,\mathrm{d}x),$$

and $(\widetilde{Y}^n(C), \widetilde{Z}^n, \widetilde{\psi}^n) \to (\widetilde{Y}(C), \widetilde{Z}, \widetilde{\psi})$ in $\mathbb{S}^\infty \times \mathbb{H}^2 \times \mathbb{J}^2$ as $n \to \infty$. Note that $\widetilde{\psi} \in \mathbb{J}^\infty$ under $\mathbb{P}$ since $\|\widetilde{\psi}\|_{\mathbb{J}^\infty} \le 2\|\widetilde{Y}(C)\|_{\mathbb{S}^\infty} < \infty$. Let $\widetilde{U}_t(C) := u_{1+1/\theta}(\exp\{\widetilde{Y}_t(C)\})$. Then, the generalized Ito formula yields

$$\begin{aligned}\widetilde{U}_t(C) &= u_{1+1/\theta}(C_T) + \int_t^T f_\theta\big(C_s, \widetilde{U}_s(C)\big)\mathrm{d}s - \int_t^T e^{-\widetilde{Y}_s(C)/\theta}\widetilde{Z}_s^\top \mathrm{d}B_s \\ &\quad - \int_t^T\int_J u_{1+1/\theta}(\exp\{\widetilde{Y}_{s-}(C)\})\Big(e^{-\widetilde{\psi}_s(x)/\theta} - 1\Big)\widetilde{\mu}(\mathrm{d}s,\mathrm{d}x),\end{aligned}$$

for any $t \in [0,T]$. Since $(\widetilde{Y}(C), \widetilde{\psi}) \in \mathbb{S}^\infty \times \mathbb{J}^\infty$ and $\widetilde{Z} \in \mathbb{H}^2$, the stochastic integrals in the above are square-integrable martingales. Hence, taking a conditional expectation, we have

$$\widetilde{U}_t(C) = \mathbb{E}_t\left[u_{1+1/\theta}(C_T) + \int_t^T f_\theta\big(C_s, \widetilde{U}_s(C)\big)\mathrm{d}s\right],$$

for any $t \in [0,T]$. Thus, from Proposition 2.5, we have $\widetilde{U}_t(C) = U_t^{\mathrm{EZ},1+1/\theta}(C)$ for any $t \in [0,T]$ a.s. on $(\Omega, \mathcal{F}, \mathbb{P})$. Finally, we obtain on $(\Omega, \mathcal{F}, \mathbb{P})$:

$$\begin{aligned}Y_t^{\mathrm{rob.},\theta}(C) &= \inf_{(\widetilde{Z},\widetilde{\psi})\in\mathbb{M}^\infty} Y_t^{(\widetilde{Z},\widetilde{\psi})}(C) \\ &\le \lim_{n\to\infty} Y_t^{(\widetilde{\mathcal{Z}}_n(\widetilde{Z}^n), \widetilde{\varphi}_n(\widetilde{\psi}^n))}(C) = \lim_{n\to\infty} \widetilde{Y}_t^n(C) = \widetilde{Y}_t(C) = \log\circ u_{1+1/\theta}^{-1}\big(U_t^{\mathrm{EZ},1+1/\theta}(C)\big).\end{aligned}$$

Hence, $u_{1+1/\theta}(\exp\{Y_t^{\mathrm{rob.},\theta}(C)\}) \le U_t^{\mathrm{EZ},1+1/\theta}(C)$ for any $t \in [0,T]$ a.s. on $(\Omega, \mathcal{F}, \mathbb{P})$. □

A.5 Proof of Proposition 2.12

Proof. We only consider the case $\gamma > 1$; the case $\gamma < 1$ follows by the same reasoning. Fix $C = (C_t)_{t\in[0,\infty)} \in \mathcal{C}^{\mathrm{E}}_{\gamma,\infty}$ arbitrarily. We first assume that C is bounded above and away from zero, and we later treat the unbounded case. For any $n \in \mathbb{N}$, define

$$\xi_n := \frac{1}{1-\gamma}\exp\left\{\mathbb{E}_n\left[\int_n^\infty \delta e^{-\delta(s-n)}\log C_s^{1-\gamma}\mathrm{d}s\right]\right\}.$$

Since C is bounded above and away from zero, $((C_t)_{t\in[0,n]}, \xi_n)$ belongs to $\mathcal{C}^{\mathrm{B}}_n$. Thus, from Proposition 2.4, for any $n \in \mathbb{N}$, there exists a finite-horizon unit EIS Epstein–Zin SDU of $((C_t)_{t\in[0,n]}, \xi_n)$ on

$[0,n]$, denoted by $U^n((C_t)_{t\in[0,n]},\xi_n)$. Furthermore, we can see that $U^n((C_t)_{t\in[0,n]},\xi_n)$ is uniformly bounded in n as follows. Let $\underline{C} := \operatorname*{ess\,inf}_{t\in[0,\infty)} C_t > 0$ and $\overline{C} = \operatorname*{ess\,sup}_{t\in[0,\infty)} C_t < \infty$. From (2.12), we have

$$\begin{aligned}
0 < \overline{C}^{1-\gamma} &\leq \exp\left\{\mathbb{E}_t\left[\int_t^\infty \delta e^{-\delta(s-t)} \log C_s^{1-\gamma} \mathrm{d}s\right]\right\} \\
&= \exp\left\{\mathbb{E}_t\left[\int_t^n \delta e^{-\delta(s-t)} \log C_s^{1-\gamma} \mathrm{d}s + e^{-\delta(n-t)} \log\big((1-\gamma)\xi_n\big)\right]\right\} \\
&\leq (1-\gamma) U_t^n\big((C_s)_{s\in[0,n]},\xi_n\big) \\
&\leq \mathbb{E}_t\left[\int_t^n \delta e^{-\delta(s-t)} C_s^{1-\gamma} \mathrm{d}s + e^{-\delta(n-t)}(1-\gamma)\xi_n\right] \leq \mathbb{E}_t\left[\int_t^\infty \delta e^{-\delta(s-t)} C_s^{1-\gamma} \mathrm{d}s\right] \leq \underline{C}^{1-\gamma} < \infty,
\end{aligned}$$

for any $n \in \mathbb{N}$ and $t \in [0,n]$. Meanwhile, for any $n \in \mathbb{N}$, (2.12) yields

$$\begin{aligned}
(1-\gamma)\xi_n &= \exp\left\{\mathbb{E}_n\left[\int_n^\infty \delta e^{-\delta(s-n)} \log C_s^{1-\gamma} \mathrm{d}s\right]\right\} \\
&= \exp\left\{\mathbb{E}_n\left[\int_n^{n+1} \delta e^{-\delta(s-n)} \log C_s^{1-\gamma} \mathrm{d}s + e^{-\delta(n+1-n)} \log\big((1-\gamma)\xi_{n+1}\big)\right]\right\} \\
&\leq (1-\gamma) U_n^{n+1}\big((C_t)_{t\in[0,n+1]},\xi_{n+1}\big).
\end{aligned}$$

Hence, $U_n^n((C_t)_{t\in[0,n]},\xi_n) = \xi_n \geq U_n^{n+1}((C_t)_{t\in[0,n+1]},\xi_{n+1})$ for any $n \in \mathbb{N}$. Thus, from Proposition 2.5, $U_t^n((C_s)_{s\in[0,n]},\xi_n) \geq U_t^{n+1}((C_s)_{s\in[0,n+1]},\xi_{n+1})$ holds for any $n \in \mathbb{N}$ and $t \in [0,n]$ a.s. Let

$$U_t(C) := \lim_{n\to\infty} U_t^n\big((C_s)_{s\in[0,n]},\xi_n\big) = \inf_{n\in\mathbb{N}} U_t^n\big((C_s)_{s\in[0,n]},\xi_n\big), \qquad t \in [0,\infty).$$

Then, $-\infty < u_\gamma(\underline{C}) \leq U_t(C) \leq u_\gamma(\overline{C}) < 0$ holds for any $t \in [0,\infty)$ a.s. Therefore, for any finite $0 \leq t \leq T < \infty$, the bounded convergence theorem yields

$$\begin{aligned}
U_t(C) = \lim_{n\to\infty} U_t^n\big((C_r)_{r\in[0,n]},\xi_n\big) &= \lim_{n\to\infty} \mathbb{E}_t\left[U_T^n\big((C_r)_{r\in[0,n]},\xi_n\big) + \int_t^T f\Big(C_s, U_s^n\big((C_r)_{r\in[0,n]},\xi_n\big)\Big)\mathrm{d}s\right] \\
&= \mathbb{E}_t\left[U_T(C) + \int_t^T f\big(C_s, U_s(C)\big)\mathrm{d}s\right].
\end{aligned}$$

Since $U_t(C) + \int_0^t f(C_s, U_s(C))\mathrm{d}s$ is a bounded martingale on $[0,\infty)$, there exists a càdlàg version M. Define

$$U_t^*(C) := M_t - \int_0^t f(C_s, U_s(C))\mathrm{d}s, \qquad t \in [0,\infty).$$

Then, $U^*(C)$ is a càdlàg modification of $U(C)$, and $U^*(C)$ also satisfies (2.19). Hence, $U^*(C)$ is an infinite-horizon unit EIS Epstein–Zin SDU of C.

We now examine the unbounded case. Fix $C \in \mathcal{C}_{\gamma,\infty}^{\mathrm{E}}$ arbitrarily. Consider the cut-off version of C, defined as in the proof of Proposition 2.4, denoted by $C^{m,k}$ with $m,k \in \mathbb{N}$. Let

$$\xi_n^{m,k} := \frac{1}{1-\gamma} \exp\left\{\mathbb{E}_n\left[\int_n^\infty \delta e^{-\delta(s-n)} \log\big((C_s^{m,k})^{1-\gamma}\big)\mathrm{d}s\right]\right\}, \qquad n,m,k \in \mathbb{N}.$$

Then for any $n, m, k \in \mathbb{N}$, we can define the finite-horizon unit EIS Epstein–Zin SDU on $[0, n]$, i.e., $U^n((C_t^{m,k})_{t\in[0,n]}, \xi_n^{m,k})$. From the argument in the bounded case, an infinite-horizon Epstein–Zin SDU of $C^{m,k}$, exists, denoted by $U(C^{m,k})$.

We now examine whether the limit of $U(C^{m,k})$ exists as $k \to \infty$ and $m \to \infty$. Applying Proposition 2.5 to the finite-horizon SDUs and letting $n \to \infty$ yields $U_t(C^{m+1,k}) \le U_t(C^{m,k}) \le U_t(C^{m,k+1})$ for any $t \in [0, \infty)$ and $m, k \in \mathbb{N}$. Furthermore, by construction,

$$\exp\left\{\mathbb{E}_t\left[\int_t^\infty \delta e^{-\delta(s-t)} \log\left((C_s^{m,k})^{1-\gamma}\right)\mathrm{d}s\right]\right\} \le (1-\gamma)U_t(C^{m,k}) \le \mathbb{E}_t\left[\int_t^\infty \delta e^{-\delta(s-t)}(C_s^{m,k})^{1-\gamma}\mathrm{d}s\right].$$

Recall $C \in \mathcal{C}^{\mathrm{E}}_{\gamma,\infty}$. Thus, by the same argument as in the proof of Proposition 2.4, there exists a non-negative and progressively measurable process K^C such that

$$\mathbb{E}_t\left[\int_t^\infty \delta e^{-\delta(s-t)}\left(\frac{C_s^{m,k}}{C_t^{m,k}}\right)^{1-\gamma}\mathrm{d}s\right] \le K_t^C + 1, \qquad t \in [0,\infty),$$

and

$$\delta e^{-1} u_\gamma(C_t \wedge 1) \le f\big(C_t^{m,k}, U_t(C^{m,k})\big) \le -\delta u_\gamma(C_t \wedge 1)(K_t^C + 1)\log(K_t^C + 1), \qquad t \in [0,\infty).$$

Thus, for any finite $T \in [0, \infty)$, $\{(f(C_t^{m,k}, U_t(C^{m,k})))_{t\in[0,T]}\}_{m,k\in\mathbb{N}}$ is uniformly integrable on $\mathrm{d}t\otimes\mathrm{d}\mathbb{P}$. Accordingly, by the same argument as in the proof of Proposition 2.4, we can define a càdlàg version of

$$U_t(C) := \inf_{m\in\mathbb{N}} \sup_{k\in\mathbb{N}} U_t(C^{m,k}), \qquad t \in [0,\infty),$$

and it satisfies (2.19). By construction, $U(C)$ satisfies (2.20). This implies that $U(C)$ is of class DL. Hence, $U(C)$ is an infinite-horizon unit EIS Epstein–Zin SDU of C. □

A.6 Proof of Proposition 2.13

Proof. We examine uniqueness in the case $\gamma > 1$. Let U^1 and U^2 be infinite-horizon unit EIS Epstein–Zin SDUs of $C \in \mathcal{C}^{\mathrm{E}}_{\gamma,\infty}$. Assume that both satisfy the inequality (2.21) for some constants $0 < \underline{K} \le \overline{K} < \infty$. Let $\Delta U_t := U_t^1(C) - U_t^2(C)$ for $t \in [0, \infty)$. Then, as in the proof of Proposition 2.5, we have

$$\Delta U_t \le \mathbb{E}_t\left[e^{\int_t^T \alpha_s \mathrm{d}s}\Delta U_T\right],$$

for any $0 \le t \le T < \infty$, where

$$\alpha_s = \frac{\partial f(C_s, U_s^1(C))}{\partial v} = -\delta - \delta\log\frac{U_s^1(C)}{u_\gamma(C_s)} = -\delta + \frac{f(C_s, U_s^1(C))}{U_s^1(C)}.$$

Since $U^1(C)$ and $U^2(C)$ satisfy the inequality (2.21), we have

$$\Delta U_t \le \mathbb{E}_t\left[e^{\int_t^T \alpha_s \mathrm{d}s}\Delta U_T\right] \le (\underline{K} - \overline{K})\mathbb{E}_t\left[\exp\left\{\int_t^T \frac{f(C_s, U_s^1(C))}{U_s^1(C)}\mathrm{d}s\right\}u_\gamma(C_T)\right]e^{-\delta(T-t)}. \qquad \text{(A.13)}$$

Meanwhile, applying the integration-by-parts formula to U^1 and taking conditional expectation gives

$$U_t^1(C) = \mathbb{E}_t\bigg[\exp\bigg\{\int_t^{\tau_n} \frac{f(C_s, U_s^1(C))}{U_s^1(C)}\mathrm{d}s\bigg\}U_{\tau_n}^1(C)\bigg],$$

where $\{\tau_n\}_{n\in\mathbb{N}}$ is a localizing sequence satisfying $\tau_n \to T$ a.s. as $n \to \infty$. Since $f(C_s, U_s^1)/U_s^1$ is bounded and U^1 is of class DL, letting $n \to \infty$ yields

$$U_t^1(C) = \mathbb{E}_t\bigg[\exp\bigg\{\int_t^{T} \frac{f(C_s, U_s^1(C))}{U_s^1(C)}\mathrm{d}s\bigg\}U_{T}^1(C)\bigg] \le \underline{K}\mathbb{E}_t\bigg[\exp\bigg\{\int_t^{T} \frac{f(C_s, U_s^1(C))}{U_s^1(C)}\mathrm{d}s\bigg\}u_\gamma(C_T)\bigg] < 0. \tag{A.14}$$

Hence, substituting (A.14) into (A.13), we have

$$\Delta U_t \le \frac{\underline{K} - \overline{K}}{\underline{K}} U_t^1(C) e^{-\delta(T-t)}. \tag{A.15}$$

The right-hand side of (A.15) converges to zero a.s. as $T \to \infty$. Thus, $\Delta U_t \le 0$. A symmetric argument gives $\Delta U_t \ge 0$, and hence, $U_t^1(C) = U_t^2(C)$ holds for all $t \in [0, \infty)$ a.s. □

A.7 Proof of Lemma 3.4

Proof. We first examine Π^*. Define

$$FOC^{\Pi}(\eta, \Pi) := m_e(\eta) - \gamma\sigma^2(\eta)\Pi + \lambda(\eta)\int_{\mathbb{R}}(1 + \Pi x)^{-\gamma} x\zeta(\mathrm{d}x), \qquad (\eta, \Pi) \in \mathbb{R}^2.$$

From conditions 4, 5, and 6 in Assumption 3.1, $FOC^{\Pi}(\eta, \Pi)$ is well defined on $\mathbb{R} \times [\underline{K}_{\Pi}, \overline{K}_{\Pi}]$. Furthermore, we have

$$\frac{\partial FOC^{\Pi}(\eta, \Pi)}{\partial \Pi} = -\gamma\sigma^2(\eta) - \gamma\lambda(\eta)\int_{\mathbb{R}}(1 + \Pi x)^{-\gamma-1}x^2\zeta(\mathrm{d}x) < 0,$$

for any $(\eta, \Pi) \in \mathbb{R} \times (\underline{K}_{\Pi}, \overline{K}_{\Pi})$. Thus, $\Pi \to FOC^{\Pi}(\eta, \Pi)$ is decreasing for any $\eta \in \mathbb{R}$. Additionally, conditions 4, 5, and 6 in Assumption 3.1 imply that for any $\eta \in \mathbb{R}$, there exists $\Pi^*(\eta) \in [\underline{K}_{\Pi}, \overline{K}_{\Pi}]$ such that $FOC^{\Pi}(\eta, \Pi^*(\eta)) = 0$ holds. Indeed, such a $\Pi^*(\eta)$ is unique. From the implicit function theorem, $\Pi^*(\eta)$ is continuously differentiable and

$$\begin{aligned}
\frac{\partial \Pi^*(\eta)}{\partial \eta} &= -\bigg(\frac{\partial FOC^{\Pi}(\eta, \Pi^*(\eta))}{\partial \Pi}\bigg)^{-1}\frac{\partial FOC^{\Pi}(\eta, \Pi^*(\eta))}{\partial \eta} \\
&= \frac{1}{\gamma}\bigg(\sigma^2(\eta) + \lambda(\eta)\int_{\mathbb{R}}(1 + \Pi^*(\eta)x)^{-\gamma-1}x^2\zeta(\mathrm{d}x)\bigg)^{-1} \\
&\quad \times \bigg(\frac{\partial m_e(\eta)}{\partial \eta} - 2\gamma\sigma(\eta)\Pi^*(\eta)\frac{\partial \sigma(\eta)}{\partial \eta} + \frac{\partial \lambda(\eta)}{\partial \eta}\int_{\mathbb{R}}(1 + \Pi^*(\eta)x)^{-\gamma}x\zeta(\mathrm{d}x)\bigg).
\end{aligned}$$

From Assumption 3.1, we can see that $\partial\Pi^*/\partial\eta$ is bounded and continuous. Thus, Π^* is Lipschitz continuous.

We next examine $\widetilde{r}$. Since r, m_e, σ, λ, and Π^* are bounded, $\widetilde{r}$ is also bounded. In addition, it is continuous and differentiable. From the envelope theorem, the derivative of $\widetilde{r}$ is

$$\begin{aligned}\frac{\partial \widetilde{r}(\eta)}{\partial \eta} = (1-\gamma)\Big(\frac{\partial r(\eta)}{\partial \eta} + \frac{\partial m_e(\eta)}{\partial \eta}\Pi^*(\eta) - \gamma\sigma(\eta)\frac{\partial \sigma(\eta)}{\partial \eta}\big(\Pi^*(\eta)\big)^2\Big)\\ + \frac{\partial \lambda(\eta)}{\partial \eta}\int_{\mathbb{R}}\Big(\big(1+\Pi^*(\eta)x\big)^{1-\gamma} - 1\Big)\zeta(\mathrm{d}x).\end{aligned}$$

From Assumption 3.1, $\partial\widetilde{r}/\partial\eta$ is bounded and continuous. Thus, $\widetilde{r}$ is Lipschitz continuous and continuously differentiable. □

A.8 Proof of Lemma 3.5

Proof. The proof follows the argument in Kraft et al. (2017). For this proof, we work on an auxiliary filtered probability space supporting a one-dimensional Brownian motion B^1, and define the diffusion η by (3.1). Since we only study the solvability of the reduced PDE (3.10), this auxiliary construction is without loss of generality.

1. Approximations. Consider the iterative PDE (3.12). The constants $\overline{h}_{\text{stab.}}$, $\underline{h}_{\text{stab.}}$, and $\underline{h}^*_{\text{stab.}}$ are defined as follows.

$$\begin{aligned}&\overline{h}_{\text{stab.}} := \left(\left(\frac{\delta e^{K_{\text{stab.}}/\delta - 1}}{\|\widetilde{r}_{\text{stab.}}\|_\infty \vee 1} + \overline{k}\right)e^{(\|\widetilde{r}_{\text{stab.}}\|_\infty \vee 1)T}\right) \vee 1,\\ &\underline{h}_{\text{stab.}} := e^{-((\|\widetilde{r}_{\text{stab.}}\|_\infty \vee 1) + \delta\log\overline{h}_{\text{stab.}})T}\overline{k}, \qquad \underline{h}^*_{\text{stab.}} := \frac{1 + e^{((\|\widetilde{r}_{\text{stab.}}\|_\infty \vee 1)+\delta\log\overline{h}_{\text{stab.}})T}}{\underline{h}_{\text{stab.}}},\end{aligned}$$

where $\|\widetilde{r}_{\text{stab.}}\|_\infty := \sup_{\eta\in\mathbb{R}}|\widetilde{r}_{\text{stab.}}(\eta)|$. Recall that

$$\widetilde{r}_{\text{stab.}}(\eta) = \widetilde{r}(\eta) - K_{\text{stab.}}, \qquad f_*(h) = -\rho_*(h)\big(\delta\log\rho_*(h) - K_{\text{stab.}}\big),$$

where ρ_* is a non-decreasing and Lipschitz continuous function from $\mathbb{R}$ to $[1/(\underline{h}^*_{\text{stab.}}+1), \overline{h}_{\text{stab.}}+1]$, as well as it has a Lipschitz-continuous derivative. Additionally, $\rho_*(x) = x$ if $x \in [1/\underline{h}^*_{\text{stab.}}, \overline{h}_{\text{stab.}}]$, $\rho_*(x) = 1/(\underline{h}^*_{\text{stab.}}+1)$ if $x \le 1/(\underline{h}^*_{\text{stab.}}+2)$, and $\rho_*(x) = \overline{h}_{\text{stab.}}+1$ if $x \ge \overline{h}_{\text{stab.}}+2$. Since f_* is bounded and Lipschitz continuous, as shown in Corollary C.2 in Kraft et al. (2017), the PDE (3.12) has a unique solution $h^n \in C_b^{1,2}([0,T]\times\mathbb{R})$ if $h^{n-1} \in C_b^{1,2}([0,T]\times\mathbb{R})$. If $n=1$, then $f_*(h^{n-1}(t,\eta)) = f_*(h^0(t,\eta)) = f_*(\overline{k})$, which is a constant and therefore bounded and Lipschitz continuous. Hence, the PDE (3.12) has a unique solution $h^1 \in C_b^{1,2}([0,T]\times\mathbb{R})$. Therefore, by mathematical induction, for all $n\in\mathbb{N}$, the PDE (3.12) has a solution $h^n \in C_b^{1,2}([0,T]\times\mathbb{R})$.

For any $(t_0,\eta) \in [0,T)\times\mathbb{R}$ and $n\in\mathbb{N}$, let $X_t^{n,(t_0,\eta)} := h^n(t,\eta_t^{(t_0,\eta)})$ for $t\in[t_0,T]$. Using the standard argument, we obtain

$$X_t^{n,(t_0,\eta)} = \overline{k} + \int_t^T \big(\widetilde{r}_{\text{stab.}}(\eta_s^{(t_0,\eta)})X_s^{n,(t_0,\eta)} + f_*(X_s^{n-1,(t_0,\eta)})\big)\mathrm{d}s - \int_t^T \beta(\eta_s^{(t_0,\eta)})\frac{\partial h^n(\eta_s^{(t_0,\eta)})}{\partial\eta}\mathrm{d}B_s^1, \tag{A.16}$$

for $t\in[t_0,T]$. Note that the stochastic integral in (A.16) is a quadratic-integrable martingale since $h^n \in C_b^{1,2}([0,T]\times\mathbb{R})$ and β is bounded.

2. Uniform bounds. We derive uniform bounds of $\{h^n\}_{n\in\mathbb{N}}$. From the optional sampling theorem, applying the integration-by-parts formula to (A.16) and taking conditional expectations yields

$$X_t^{n,(t_0,\eta)} = \mathbb{E}_t\bigg[\int_t^T e^{\int_t^s \widetilde{r}_{\text{stab.}}(\eta_u^{(t_0,\eta)})\mathrm{d}u} f_*(X_s^{n-1,(t_0,\eta)})\mathrm{d}s + e^{\int_t^T \widetilde{r}_{\text{stab.}}(\eta_s^{(t_0,\eta)})\mathrm{d}s}\overline{k}\bigg], \qquad t\in[t_0,T].$$

Since $f_*(x) \le \delta e^{K_{\text{stab.}}/\delta-1}$ for any $x>0$, we have

$$\begin{aligned} X_t^{n,(t_0,\eta)} &\le \mathbb{E}_t\bigg[\delta e^{K_{\text{stab.}}/\delta-1}\int_t^T e^{\int_t^s \widetilde{r}_{\text{stab.}}(\eta_u^{(t_0,\eta)})\mathrm{d}u}\mathrm{d}s + e^{\int_t^T \widetilde{r}_{\text{stab.}}(\eta_s^{(t_0,\eta)})\mathrm{d}s}\overline{k}\bigg] \\ &\le \frac{\delta e^{K_{\text{stab.}}/\delta-1}}{\|\widetilde{r}_{\text{stab.}}\|_\infty \vee 1}\Big(e^{(\|\widetilde{r}_{\text{stab.}}\|_\infty\vee 1)(T-t)}-1\Big) + \overline{k}e^{(\|\widetilde{r}_{\text{stab.}}\|_\infty\vee 1)(T-t)} \\ &\le \bigg(\frac{\delta e^{K_{\text{stab.}}/\delta-1}}{\|\widetilde{r}_{\text{stab.}}\|_\infty \vee 1}+\overline{k}\bigg)e^{(\|\widetilde{r}_{\text{stab.}}\|_\infty\vee 1)T} \le \overline{h}_{\text{stab.}}, \end{aligned}$$

for any $t\in[t_0,T]$. This implies $X_t^{n,(t_0,\eta)} \le \overline{h}_{\text{stab.}}$ for any $t\in[t_0,T]$.

Here, we have

$$f_*(h) \ge -\delta(\log \overline{h}_{\text{stab.}})\overline{h}_{\text{stab.}}, \quad h\in(-\infty,\overline{h}_{\text{stab.}}]$$

Therefore,

$$\begin{aligned} X_t^{n,(t_0,\eta)} &\ge \mathbb{E}_t\bigg[-\delta(\log \overline{h}_{\text{stab.}})\overline{h}_{\text{stab.}}\int_t^T e^{\int_t^s \widetilde{r}_{\text{stab.}}(\eta_u^{(t_0,\eta)})\mathrm{d}u}\mathrm{d}s + e^{\int_t^T \widetilde{r}_{\text{stab.}}(\eta_s^{(t_0,\eta)})\mathrm{d}s}\overline{k}\bigg] \\ &\ge -\delta(\log \overline{h}_{\text{stab.}})\overline{h}_{\text{stab.}}\frac{1}{\|\widetilde{r}_{\text{stab.}}\|_\infty\vee 1}e^{(\|\widetilde{r}_{\text{stab.}}\|_\infty\vee 1)T} + e^{-(\|\widetilde{r}_{\text{stab.}}\|_\infty\vee 1)T}\overline{k} =: \underline{h}^{(1)}. \end{aligned}$$

This implies $X_t^{n,(t_0,\eta)} \ge \underline{h}^{(1)}$ for any $t\in[t_0,T]$.

3. The fixed point argument and a tighter lower bound. From step 2, we have $\underline{h}^{(1)} \le X_t^{n,(t_0,\eta)} \le \overline{h}_{\text{stab.}}$ for any $(t,n)\in[t_0,T]\times\mathbb{N}$. Accordingly, we have

$$\big|f_*(X_s^{n,(t_0,\eta)}) - f_*(X_s^{n-1,(t_0,\eta)})\big| \le \underbrace{\sup_{x\in[\underline{h}^{(1)},\overline{h}_{\text{stab.}}]}\Big\{\delta|1+\log\rho_*(x)||\rho_*'(x)|+K_{\text{stab.}}\Big\}}_{=:c}|X_s^{n,(t_0,\eta)} - X_s^{n-1,(t_0,\eta)}|,$$

for any $(s,n)\in[t_0,T]\times\mathbb{N}$. Therefore, from the optional sampling theorem, we have

$$\begin{aligned} |X_t^{n+1,(t_0,\eta)} - X_t^{n,(t_0,\eta)}| &\le \mathbb{E}_t\bigg[\int_t^T e^{\int_t^s \widetilde{r}_{\text{stab.}}(\eta_u^{(t_0,\eta)})\mathrm{d}u}\big|f_*(X_s^{n,(t_0,\eta)}) - f_*(X_s^{n-1,(t_0,\eta)})\big|\mathrm{d}s\bigg] \\ &\le c\int_t^T e^{(\|\widetilde{r}_{\text{stab.}}\|_\infty\vee 1)(s-t)}\mathbb{E}_t\big[|X_s^{n,(t_0,\eta)} - X_s^{n-1,(t_0,\eta)}|\big]\mathrm{d}s, \end{aligned}$$

for any $(t,n)\in[t_0,T]\times\mathbb{N}$. Since $\{X^{n,(t_0,\eta)}\}_{n\in\mathbb{N}}$ is uniformly bounded, from Proposition 6.4 in Kraft et al. (2017), there exists the limit of $\{X^{n,(t_0,\eta)}\}_{n\in\mathbb{N}}$, denoted by $X^{*,(t_0,\eta)}$, such that

$$\operatorname*{ess\,sup}_{t\in[t_0,T]} |X_t^{n,(t_0,\eta)} - X_t^{*,(t_0,\eta)}| \le e^{(\|\widetilde{r}_{\mathrm{stab.}}\|_\infty \vee 1)T}\big(\overline{k} + \overline{h}_{\mathrm{stab.}} \vee |\underline{h}^{(1)}|\big)\left(\frac{ecT}{n}\right)^n,$$

holds for any $n > cT$. Let

$$h(t_0,\eta) := X_{t_0}^{*,(t_0,\eta)} = \lim_{n\to\infty} X_{t_0}^{n,(t_0,\eta)} = \lim_{n\to\infty} h^n(t_0,\eta).$$

Since $\|\widetilde{r}_{\mathrm{stab.}}\|_\infty$ and c are independent of (t_0,η), we have

$$\begin{aligned}|h^n(t_0,\eta) - h(t_0,\eta)| &= |X_{t_0}^{n,(t_0,\eta)} - X_{t_0}^{*,(t_0,\eta)}| \\ &\le \operatorname*{ess\,sup}_{t\in[t_0,T]} |X_t^{n,(t_0,\eta)} - X_t^{*,(t_0,\eta)}| \le e^{(\|\widetilde{r}_{\mathrm{stab.}}\|_\infty \vee 1)T}\big(\overline{k} + \overline{h}_{\mathrm{stab.}} \vee |\underline{h}^{(1)}|\big)\left(\frac{ecT}{n}\right)^n\end{aligned}$$

holds for any $(t_0,\eta) \in [0,T]\times\mathbb{R}$ and $n > cT$. Therefore, we have

$$\|h^n - h\|_\infty \le e^{(\|\widetilde{r}_{\mathrm{stab.}}\|_\infty \vee 1)T}\big(\overline{k} + \overline{h}_{\mathrm{stab.}} \vee |\underline{h}^{(1)}|\big)\left(\frac{ecT}{n}\right)^n,$$

for any $n > cT$. Thus, $\{h^n\}_{n\in\mathbb{N}}$ uniformly converges to h and h is continuous.

We now derive a tighter lower bound of h. Since $\rho_*(h) \le (h\vee 0) + (\underline{h}^*_{\mathrm{stab.}})^{-1}$ for all $h\in\mathbb{R}$, we have

$$f_*(h) \ge -\delta(\log \overline{h}_{\mathrm{stab.}})\rho_*(h) \ge -\delta(\log \overline{h}_{\mathrm{stab.}})\big((h\vee 0) + (\underline{h}^*_{\mathrm{stab.}})^{-1}\big) \tag{A.17}$$

for any $h \le \overline{h}_{\mathrm{stab.}}$. Let $\delta_t := \delta(\log \overline{h}_{\mathrm{stab.}})\mathbb{1}\{X_t^{*,(t_0,\eta)} \ge 0\}$ for $t\in[t_0,T]$. From the optional sampling theorem and bounded convergence theorem, we have

$$X_t^{*,(t_0,\eta)} = \mathbb{E}_t\left[\int_t^T e^{\int_t^s \left(\widetilde{r}_{\mathrm{stab.}}(\eta_u^{(t_0,\eta)}) - \delta_u\right)\mathrm{d}u}\big(f_*(X_s^{*,(t_0,\eta)}) + \delta_s X_s^{*,(t_0,\eta)}\big)\mathrm{d}s + e^{\int_t^T \left(\widetilde{r}_{\mathrm{stab.}}(\eta_s^{(t_0,\eta)}) - \delta_s\right)\mathrm{d}s}\overline{k}\right],$$

for any $t\in[t_0,T]$. Since $X_s^{*,(t_0,\eta)} \le \overline{h}_{\mathrm{stab.}}$ holds for any $s\in[t_0,T]$ a.s. and f_* satisfies the inequality (A.17), we have

$$\begin{aligned}X_t^{*,(t_0,\eta)} &\ge \mathbb{E}_t\left[-\frac{\delta(\log \overline{h}_{\mathrm{stab.}})}{\underline{h}^*_{\mathrm{stab.}}}\int_t^T e^{\int_t^s \left(\widetilde{r}_{\mathrm{stab.}}(\eta_u^{(t_0,\eta)}) - \delta_u\right)\mathrm{d}u}\mathrm{d}s + e^{\int_t^T \left(\widetilde{r}_{\mathrm{stab.}}(\eta_s^{(t_0,\eta)}) - \delta_s\right)\mathrm{d}s}\overline{k}\right] \\ &\ge -\frac{\delta(\log \overline{h}_{\mathrm{stab.}})}{\underline{h}^*_{\mathrm{stab.}}}\int_t^T e^{\int_t^s \left((\|\widetilde{r}_{\mathrm{stab.}}\|_\infty\vee 1) + \delta\log \overline{h}_{\mathrm{stab.}}\right)\mathrm{d}u}\mathrm{d}s + e^{-\int_t^T \left((\|\widetilde{r}_{\mathrm{stab.}}\|_\infty\vee 1) + \delta\log \overline{h}_{\mathrm{stab.}}\right)\mathrm{d}s}\overline{k} \\ &\ge -\frac{\delta(\log \overline{h}_{\mathrm{stab.}})}{\underline{h}^*_{\mathrm{stab.}}((\|\widetilde{r}_{\mathrm{stab.}}\|_\infty \vee 1) + \delta\log \overline{h}_{\mathrm{stab.}})}e^{((\|\widetilde{r}_{\mathrm{stab.}}\|_\infty\vee 1) + \delta\log \overline{h}_{\mathrm{stab.}})T} + e^{-((\|\widetilde{r}_{\mathrm{stab.}}\|_\infty\vee 1) + \delta\log \overline{h}_{\mathrm{stab.}})T}\overline{k} \\ &\ge -\frac{1}{\underline{h}^*_{\mathrm{stab.}}}e^{((\|\widetilde{r}_{\mathrm{stab.}}\|_\infty\vee 1) + \delta\log \overline{h}_{\mathrm{stab.}})T} + \underline{h}_{\mathrm{stab.}} = \frac{1}{\underline{h}^*_{\mathrm{stab.}}} > 0,\end{aligned}$$

for any $t\in[t_0,T]$, where the last equality is due to the definition of $\underline{h}^*_{\mathrm{stab.}}$. Thus, $X_t^{*,(t_0,\eta)} \ge 1/\underline{h}^*_{\mathrm{stab.}}$

holds for any $t \in [t_0, T]$, and hence $\rho^*(X_t^{*,(t_0,\eta)}) = X_t^{*,(t_0,\eta)}$ holds for any $t \in [t_0, T]$ a.s. This implies

$$f_*(X_t^{*,(t_0,\eta)}) = -X_t^{*,(t_0,\eta)}\big(\delta \log X_t^{*,(t_0,\eta)} - K_{\text{stab.}}\big) \geq -\delta(\log \overline{h}_{\text{stab.}})X_t^{*,(t_0,\eta)}, \qquad t \in [t_0, T].$$

Therefore, from the same argument, we have

$$X_t^{*,(t_0,\eta)} \geq e^{-((\|\widetilde{r}_{\text{stab.}}\|_\infty \vee 1)+\delta \log \overline{h}_{\text{stab.}})T}\overline{k} = \underline{h}_{\text{stab.}} > 0, \qquad t \in [t_0, T].$$

This implies $h(t_0, \eta) \geq \underline{h}_{\text{stab.}}$ for any $(t_0, \eta) \in [0, T] \times \mathbb{R}$.

4. The regularity of the limit function. We examine the differentiability of h. For any $t_0 \in [0, T)$ and $\eta \in \mathbb{R}$, let $X_t^{*,(t_0,\eta)} := h(t, \eta_t^{(t_0,\eta)})$ for $t \in [t_0, T]$. Then, from the bounded convergence theorem, we obtain

$$\begin{aligned} X_t^{*,(t_0,\eta)} &= h(t, \eta_t^{(t_0,\eta)}) = \lim_{n\to\infty} h^n(t, \eta_t^{(t_0,\eta)}) = \lim_{n\to\infty} X_t^{n,(t_0,\eta)} \\ &= \lim_{n\to\infty} \mathbb{E}_t\bigg[\int_t^T e^{\int_t^s \widetilde{r}_{\text{stab.}}(\eta_u^{(t_0,\eta)})\mathrm{d}u} f_*(X_s^{n-1,(t_0,\eta)})\mathrm{d}s + e^{\int_t^T \widetilde{r}_{\text{stab.}}(\eta_s^{(t_0,\eta)})\mathrm{d}s}\overline{k}\bigg] \\ &= \mathbb{E}_t\bigg[\int_t^T e^{\int_t^s \widetilde{r}_{\text{stab.}}(\eta_u^{(t_0,\eta)})\mathrm{d}u} f_*(X_s^{*,(t_0,\eta)})\mathrm{d}s + e^{\int_t^T \widetilde{r}_{\text{stab.}}(\eta_s^{(t_0,\eta)})\mathrm{d}s}\overline{k}\bigg], \qquad t \in [t_0, T]. \end{aligned}$$

Therefore, from the martingale representation theorem for Brownian motions, $X^{*,(t_0,\eta)}$ is a solution to the following BSDE on $[t_0, T]$:

$$\mathrm{d}X_t^{*,(t_0,\eta)} = -\Big(\widetilde{r}_{\text{stab.}}(\eta_t^{(t_0,\eta)})X_t^{*,(t_0,\eta)} + f_*(X_t^{*,(t_0,\eta)})\Big)\mathrm{d}t + Z_t^*\mathrm{d}B_t^1, \tag{A.18}$$

with $X_T^{*,(t_0,\eta)} = \overline{k}$, where Z^* is a progressively measurable process satisfying $\mathbb{E}[\int_{t_0}^T |Z_t^*|^2\mathrm{d}t] < \infty$.

Meanwhile, consider a PDE for $\widehat{h}$ on $[0, T] \times \mathbb{R}$ such that

$$\begin{aligned} &\frac{\partial \widehat{h}(t,\eta)}{\partial t} + \widetilde{r}_{\text{stab.}}(\eta)\widehat{h}(t,\eta) + \alpha(\eta)\frac{\partial \widehat{h}(t,\eta)}{\partial \eta} + \frac{1}{2}\beta^2(\eta)\frac{\partial^2 \widehat{h}(t,\eta)}{\partial \eta^2} + f_*(\widehat{h}(t,\eta)) \\ &= \frac{\partial \widehat{h}(t,\eta)}{\partial t} + \widetilde{r}(\eta)\widehat{h}(t,\eta) + \alpha(\eta)\frac{\partial \widehat{h}(t,\eta)}{\partial \eta} + \frac{1}{2}\beta^2(\eta)\frac{\partial^2 \widehat{h}(t,\eta)}{\partial \eta^2} - \delta\rho_*\big(\widehat{h}(t,\eta)\big)\log \rho_*\big(\widehat{h}(t,\eta)\big) = 0, \end{aligned} \tag{A.19}$$

with $\widehat{h}(T, \eta) = \overline{k}$. Since f_* is bounded and Lipschitz continuous, there exists a unique solution to the PDE (A.19) in $C_b^{1,2}([0, T] \times \mathbb{R})$ (see Corollary C.4 in Kraft et al. (2017)). Let $\widehat{X}_t^{(t_0,\eta)} := \widehat{h}(t, \eta_t^{(t_0,\eta)})$ for $t \in [t_0, T]$. Then, from the PDE (A.19) and the Ito formula, we have

$$\mathrm{d}\widehat{X}_t^{(t_0,\eta)} = -\Big(\widetilde{r}_{\text{stab.}}(\eta_t^{(t_0,\eta)})\widehat{X}_t^{(t_0,\eta)} + f_*(\widehat{X}_t^{(t_0,\eta)})\Big)\mathrm{d}t + \widehat{Z}_t\mathrm{d}B_t^1, \qquad t \in [t_0, T], \tag{A.20}$$

with $\widehat{X}_T^{(t_0,\eta)} = \overline{k}$, where $\widehat{Z}_t = \beta(\eta_t^{(t_0,\eta)})\frac{\partial}{\partial \eta}\widehat{h}(t, \eta_t^{(t_0,\eta)})$ for $t \in [t_0, T]$. Note that $\widehat{Z}$ is bounded since $\widehat{h} \in C_b^{1,2}([0, T] \times \mathbb{R})$ and β is bounded. Since f_* is Lipschitz, the comparison theorem for the Lipschitz Brownian BSDEs (A.18) and (A.20) implies $\widehat{X}_t^{(t_0,\eta)} = X_t^{*,(t_0,\eta)}$ for any $t \in [t_0, T]$ a.s. Therefore, $\widehat{h}(t_0, \eta) = h(t_0, \eta)$ holds. Since (t_0, η) is arbitrary, h is a $C_b^{1,2}([0, T] \times \mathbb{R})$-function satisfying the PDE (A.19). Finally, since $0 < \underline{h}_{\text{stab.}} \leq h \leq \overline{h}_{\text{stab.}} < \infty$, we have

$$-\delta\rho_*\big(h(t,\eta)\big)\log\rho_*\big(h(t,\eta)\big) = -\delta h(t,\eta)\log\big(h(t,\eta)\big), \qquad (t,\eta)\in[0,T]\times\mathbb{R}.$$

Thus, h is a solution to the reduced HJB equation (3.10). Note that the preceding discussion holds for all $K_{\text{stab.}} \geq 0$, and that $\widehat{h}(=h)$ does not depend on $K_{\text{stab.}}$. Consequently, we may establish uniform bounds for the solution by considering the case $K_{\text{stab.}} = 0$. Let

$$\overline{h} := \left(\left(\frac{\delta}{e(\|\widetilde{r}\|_\infty \vee 1)} + \overline{k}\right)e^{(\|\widetilde{r}\|_\infty\vee 1)T}\right)\vee 1, \qquad \underline{h} := e^{-((\|\widetilde{r}\|_\infty\vee 1)+\delta\log\overline{h})T}\overline{k} = e^{-(\|\widetilde{r}\|_\infty\vee 1)T}\overline{h}^{-\delta T}\overline{k}.$$

Then, we obtain the following bounds:

$$0 < \underline{h} \leq h(t,\eta) \leq \overline{h} < \infty, \qquad (t,\eta)\in[0,T]\times\mathbb{R}.$$

□

A.9 Proof of Proposition 3.6

Proof. Fix $(t_0,w,\eta)\in[0,T]\times(0,\infty)\times\mathbb{R}$ and $(C,\Pi)\in\mathcal{A}(t_0,w,\eta)$ arbitrarily. Let

$$\begin{aligned}
v_t &:= v(t, W_t^{(t_0,w,\eta);(C,\Pi)}, \eta_t^{(t_0,\eta)}),\\
Z_t^v &:= \begin{pmatrix} \frac{\partial v(t,W_t^{(t_0,w,\eta);(C,\Pi)},\eta_t^{(t_0,\eta)})}{\partial w} W_t^{(t_0,w,\eta);(C,\Pi)}\Pi_t\sigma(\eta_t^{(t_0,\eta)}) \\ \frac{\partial v(t,W_t^{(t_0,w,\eta);(C,\Pi)},\eta_t^{(t_0,\eta)})}{\partial \eta}\beta(\eta_t^{(t_0,\eta)}) \end{pmatrix}\\
&= \big(W_t^{(t_0,w,\eta);(C,\Pi)}\big)^{1-\gamma}\begin{pmatrix} h(t,\eta_t^{(t_0,\eta)})\Pi_t\sigma(\eta_t^{(t_0,\eta)}) \\ \frac{1}{1-\gamma}\frac{\partial h(t,\eta_t^{(t_0,\eta)})}{\partial\eta}\beta(\eta_t^{(t_0,\eta)}) \end{pmatrix},\\
\psi_t^v(x) &:= v\big(t-, W_{t-}^{(t_0,w,\eta);(C,\Pi)}(1+\Pi_{t-}x), \eta_{t-}^{(t_0,\eta)}\big) - v\big(t-, W_{t-}^{(t_0,w,\eta);(C,\Pi)}, \eta_{t-}^{(t_0,\eta)}\big)\\
&= \big(W_{t-}^{(t_0,w,\eta);(C,\Pi)}\big)^{1-\gamma} h(t-,\eta_{t-}^{(t_0,\eta)})\left(\frac{(1+\Pi_{t-}x)^{1-\gamma}-1}{1-\gamma}\right)
\end{aligned}$$

for $t\in[t_0,T]$. Then, from Definition 3.2 and Lemma 3.5, there exists a finite constant $K_1 \geq 0$ such that

$$\mathbb{E}\left[\int_{t_0}^T |Z_t^v|^2\mathrm{d}t\right] \leq K_1 T\mathbb{E}\left[\sup_{t\in[t_0,T]}\big(W_t^{(t_0,w,\eta);(C,\Pi)}\big)^{2(1-\gamma)}\right] < \infty. \tag{A.21}$$

Similarly, from Definition 3.2 and Lemma 3.5, there exists a finite constant $K_2 \geq 0$ such that

$$\mathbb{E}\left[\left|\int_{t_0}^T\int_{\mathbb{R}}\psi_t^v(x)\widetilde{\mu}(\mathrm{d}t,\mathrm{d}x)\right|\right] \leq K_2 T\mathbb{E}\left[\sup_{t\in[t_0,T]}\big(W_t^{(t_0,w,\eta);(C,\Pi)}\big)^{2(1-\gamma)}\right] < \infty. \tag{A.22}$$

From the generalized Ito formula, we have

$$
\begin{aligned}
v_t &= v_T - \int_t^T \left(\left(\frac{\partial}{\partial t} + \mathcal{L}^{C_s,\Pi_s}\right) v(s, W_s^{(t_0,w,\eta);(C,\Pi)}, \eta_s^{(t_0,\eta)})\right)\mathrm{d}s \\
&\quad - \int_t^T (Z_s^v)^\top \mathrm{d}B_s - \int_t^T \int_{\mathbb{R}} \psi_s^v(x)\widetilde{\mu}(\mathrm{d}s,\mathrm{d}x) \\
&= v_T + \int_t^T \left(HJB_s^{C_s,\Pi_s} + f(C_s,v_s)\right)\mathrm{d}s - \int_t^T (Z_s^v)^\top \mathrm{d}B_s - \int_t^T \int_{\mathbb{R}} \psi_s^v(x)\widetilde{\mu}(\mathrm{d}s,\mathrm{d}x),
\end{aligned}
$$

for $t \in [t_0, T]$, where

$$
HJB_s^{C_s,\Pi_s} := -\left(\frac{\partial}{\partial t} + \mathcal{L}^{C_s,\Pi_s}\right) v(s, W_s^{(t_0,w,\eta);(C,\Pi)}, \eta_s^{(t_0,\eta)}) - f(C_s, v_s), \qquad s \in [t_0, T].
$$

From the HJB equation (3.8), $HJB_s^{C_s,\Pi_s} \geq 0$ holds for any $s \in [t_0,T]$ a.s. Furthermore, from (A.21) and (A.22), the stochastic integrals are true martingales. Thus,

$$
v_t + \int_{t_0}^t f(C_s, v_s)\mathrm{d}s, \qquad t \in [t_0, T]
$$

is a supermartingale. Furthermore, we have

$$
\mathbb{E}\left[\sup_{t\in[t_0,T]} |v_t|\right] \leq \frac{\overline{h}}{|1-\gamma|}\mathbb{E}\left[\sup_{t\in[t_0,T]} \left(W_t^{(t_0,w,\eta);(C,\Pi)}\right)^{1-\gamma}\right] < \infty.
$$

Thus, $(v_t)_{t\in[t_0,T]}$ is uniformly integrable. Meanwhile, since $(C, \overline{k}u_\gamma(W_T^{(t_0,w,\eta);(C,\Pi)})) \in \mathcal{C}_{\gamma,T}^{\mathrm{U}}$,

$$
U_t\left(C, \overline{k}u_\gamma(W_T^{(t_0,w,\eta);(C,\Pi)})\right) + \int_{t_0}^t f\left(C_s, U_s\left(C, \overline{k}u_\gamma(W_T^{(t_0,w,\eta);(C,\Pi)})\right)\right)\mathrm{d}s, \qquad t \in [t_0, T],
$$

is a local martingale and $(U_t(C, \overline{k}u_\gamma(W_T^{(t_0,w,\eta);(C,\Pi)})))_{t\in[t_0,T]}$ is uniformly integrable. In the case $\gamma < 1$, from the 4th condition of Definition 3.2, we have $v_t \geq l^{1-\gamma}\underline{h}u_\gamma(C_t)$ for any $t \in [t_0, T]$ a.s. Thus, from Proposition 2.5, $v_t \geq U_t(C, \overline{k}u_\gamma(W_T^{(t_0,w,\eta);(C,\Pi)}))$ holds for any $t \in [t_0,T]$ a.s. In the case $\gamma > 1$, since $(C, \overline{k}u_\gamma(W_T^{(t_0,w,\eta);(C,\Pi)})) \in \mathcal{C}_{\gamma,T}^{\mathrm{U}}$ holds, it satisfies the inequality (2.13). This implies that there exists a constant $K' > 0$ such that $U_t(C, \overline{k}u_\gamma(W_T^{(t_0,w,\eta);(C,\Pi)})) \leq K'u_\gamma(C_t)$ for all $t \in [t_0, T]$. Hence, Proposition 2.5 yields $v_t \geq U_t(C, \overline{k}u_\gamma(W_T^{(t_0,w,\eta);(C,\Pi)}))$ for any $t \in [t_0,T]$ a.s. In both cases $\gamma > 1$ and $\gamma < 1$, since (C, Π) and (t_0, w, η) are arbitrary, we have

$$
v(t_0,w,\eta) \geq V(t_0,w,\eta) = \sup_{(C,\Pi)\in\mathcal{A}(t_0,w,\eta)} U_{t_0}\left(C, \overline{k}u_\gamma(W_T^{(t_0,w,\eta);(C,\Pi)})\right), \quad (t_0,w,\eta) \in [0,T]\times(0,\infty)\times\mathbb{R}.
$$

We show the admissibility of (C^*, Π^*). Note that $\underline{K}_\Pi \leq \Pi^*(\eta) \leq \overline{K}_\Pi$ holds for any $\eta \in \mathbb{R}$. Since r, m_e, Π^*, and σ are bounded, the coefficients of (3.14) are Lipschitz and of linear growth in

$W^{*,(t_0,w,\eta)}$. Hence, there exists a strong solution to (3.14). The solution has the following form.

$$W_t^{*,(t_0,w,\eta)} = w\exp\left\{\int_{t_0}^t r_s^{*,(t_0,\eta)}\mathrm{d}s\right\}\mathcal{E}_t^{t_0}(M^{*,(t_0,\eta)}),$$
$$r_t^{*,(t_0,\eta)} := r(\eta_t^{(t_0,\eta)}) + m_e(\eta_t^{(t_0,\eta)})\Pi^*(\eta_t^{(t_0,\eta)}) - \delta + \lambda(\eta_t^{(t_0,\eta)})\int_{\mathbb{R}}\Pi^*(\eta_t^{(t_0,\eta)})x\zeta(\mathrm{d}x),$$
$$M_t^{*,(t_0,\eta)} := \int_{t_0}^t \Pi^*(\eta_s^{(t_0,\eta)})\sigma(\eta_s^{(t_0,\eta)})\mathrm{d}B_s^2 + \int_{t_0}^t\int_{\mathbb{R}}\Pi^*(\eta_s^{(t_0,\eta)})x\widetilde{\mu}(\mathrm{d}s,\mathrm{d}x),$$

for all $t\in[t_0,T]$, where $\mathcal{E}^{t_0}(M^{*,(t_0,\eta)})$ is the stochastic exponential of $M^{*,(t_0,\eta)}$ on $[t_0,T]$. Since σ and Π^* are bounded and there exists a constant $\widehat{\epsilon}\in(0,1)$ such that $\Pi^*(\eta)x\geq -1+\widehat{\epsilon}$ holds for any $\eta\in\mathbb{R}$ and $x\in\mathrm{supp}(\zeta)$, $\mathcal{E}^{t_0}(M^{*,(t_0,\eta)})$ is strictly positive from Proposition 2.8. Thus, $W^{*,(t_0,w,\eta)}$ takes values in $(0,\infty)$.

We prove the inequality (3.6). Let $X_t^* := |W_t^{*,(t_0,w,\eta)}|^{2(1-\gamma)}$ for $t\in[t_0,T]$. Then, from the generalized Ito formula, we obtain

$$\mathrm{d}X_t^* = X_t^*\left[\Big((1-\gamma)\big(2a_t + (1-2\gamma)b_t^2\big) + d_t\Big)\mathrm{d}t + 2(1-\gamma)b_t\mathrm{d}B_t\right] + X_{t-}^*\int_{\mathbb{R}}\psi_t^*(x)\widetilde{\mu}(\mathrm{d}t,\mathrm{d}x),$$

for $t\in[t_0,T]$, where

$$a_t := r(\eta_t^{(t_0,\eta)}) + m_e(\eta_t^{(t_0,\eta)})\Pi^*(\eta_t^{(t_0,\eta)}) - \delta, \qquad b_t := \Pi^*(\eta_t^{(t_0,\eta)})\sigma(\eta_t^{(t_0,\eta)}),$$
$$\psi_t^*(x) := \big(1+\Pi^*(\eta_t^{(t_0,\eta)})x\big)^{2(1-\gamma)} - 1, \qquad d_t := \lambda(\eta_t^{(t_0,\eta)})\int_{\mathbb{R}}\psi_t^*(x)\zeta(\mathrm{d}x).$$

We see that a_t, b_t, and λ are bounded and that $\int_{\mathbb{R}}(\psi_t^*(x))^2\zeta(\mathrm{d}x)$ is uniformly bounded in $(t,\omega)\in[t_0,T]\times\Omega$ from (3.5). Therefore, we can apply the standard argument: we first localize X^* and derive a Gronwall-type integral inequality for the expected value of localized X^*, then apply the Gronwall inequality and the Fatou lemma, yielding $\sup_{t\in[t_0,T]}\mathbb{E}[X_t^*]<\infty$. Next, we again localize X^*, take the supremum of it, and then take the expectation. Let $X_t^{*,n}$ is the stopped process $X_{t\wedge\tau_n}^*$ by $\tau_n := \inf\{t\geq t_0 \;:\; X_t^*\geq n\}\wedge T$. Applying the Burkholder–Davis–Gundy inequality to the continuous part of the martingale term and a Bichteler–Jacod type maximal inequality to the purely discontinuous part (see Marinelli and Röckner (2014)), we obtain an estimate in expectation for the supremum of the martingale term in terms of the square root of its predictable quadratic variation. Using the bound $(X_t^{*,n})^2\leq(\sup_{s\in[t_0,T]}X_s^{*,n})X_t^{*,n}$ inside the predictable quadratic variation of the martingale term and then applying the Young inequality, we obtain

$$\mathbb{E}\left[\sup_{t\in[t_0,T]}X_t^{*,n}\right]\leq K_3\left(X_{t_0}^* + \int_{t_0}^T\mathbb{E}[X_t^{*,n}]\mathrm{d}t\right)\leq K_3\left(w^{2(1-\gamma)} + T\sup_{t\in[t_0,T]}\mathbb{E}[X_t^*]\right)<\infty,$$

for all $n\in\mathbb{N}$, where $K_3>0$ is a constant independent of n. Hence, from the monotone convergence theorem, we conclude that

$$\mathbb{E}\left[\sup_{t\in[t_0,T]}|W_t^{*,(t_0,w,\eta)}|^{2(1-\gamma)}\right] = \mathbb{E}\left[\sup_{t\in[t_0,T]}X_t^*\right]<\infty.$$

We prove $(C^*(W^{*,(t_0,w,\eta)}), \overline{k}u_\gamma(W_T^{*,(t_0,w,\eta)})) \in \mathcal{C}^{\mathrm{U}}_{\gamma,T}$. We have

$$\begin{aligned}
&\mathbb{E}_t\left[\int_t^T \delta e^{-\delta(s-t)}\left(\frac{C^*(W_s^{*,(t_0,w,\eta)})}{C^*(W_t^{*,(t_0,w,\eta)})}\right)^{1-\gamma}\mathrm{d}s + e^{-\delta(T-t)}\frac{(1-\gamma)\overline{k}u_\gamma(W_T^{*,(t_0,w,\eta)})}{(C^*(W_t^{*,(t_0,w,\eta)}))^{1-\gamma}}\right]\\
&= \mathbb{E}_t\left[\int_t^T \delta e^{-\delta(s-t)}\left(\frac{W_s^{*,(t_0,w,\eta)}}{W_t^{*,(t_0,w,\eta)}}\right)^{1-\gamma}\mathrm{d}s + e^{-\delta(T-t)}\frac{\overline{k}}{\delta^{1-\gamma}}\left(\frac{W_T^{*,(t_0,w,\eta)}}{W_t^{*,(t_0,w,\eta)}}\right)^{1-\gamma}\right]\\
&= \mathbb{E}_t\left[\int_t^T \delta e^{-\delta(s-t)}\widetilde{W}_{t,s}^{1-\gamma}\mathrm{d}s + e^{-\delta(T-t)}\overline{k}\delta^{\gamma-1}\widetilde{W}_{t,T}^{1-\gamma}\right],
\end{aligned}$$

for all $t \in [t_0, T]$, where

$$\widetilde{W}_{t,s} := \frac{W_s^{*,(t_0,w,\eta)}}{W_t^{*,(t_0,w,\eta)}} = \exp\left\{\int_t^s r_u^{*,(t_0,\eta)}\mathrm{d}u\right\}\mathcal{E}_s^t(M^{*,(t_0,\eta)}), \qquad s \in [t, T].$$

From the same argument as that in the case of X^*, it can be seen that there exists a positive constant K_4 such that $\mathbb{E}_t[\sup_{s\in[t,T]} |\widetilde{W}_{t,s}|^{1-\gamma}] \le K_4$ holds for any $t \in [t_0, T]$. Thus, we have

$$\begin{aligned}
&\mathbb{E}_t\left[\int_t^T \delta e^{-\delta(s-t)}\left(\frac{C^*(W_s^{*,(t_0,w,\eta)})}{C^*(W_t^{*,(t_0,w,\eta)})}\right)^{1-\gamma}\mathrm{d}s + e^{-\delta(T-t)}\frac{(1-\gamma)\overline{k}u_\gamma(W_T^{*,(t_0,w,\eta)})}{(C^*(W_t^{*,(t_0,w,\eta)}))^{1-\gamma}}\right]\\
&\le K_4\big(1 - e^{-\delta(T-t)} + e^{-\delta(T-t)}\overline{k}\delta^{\gamma-1}\big) \le K_4(1 + \overline{k}\delta^{\gamma-1}),
\end{aligned}$$

for all $t \in [t_0, T]$. Thus, the inequality (2.9) holds for some constant $K_5 > 0$. This also implies that the inequality (2.7) holds, when $t = t_0$ and taking the unconditional expectation. Meanwhile, we have

$$\begin{aligned}
&\mathbb{E}\left[\int_{t_0}^T \left(1 \vee \left(C^*(W_s^{*,(t_0,w,\eta)})\right)^{1-\gamma}\right)(K_5+1)\log(K_5+1)\mathrm{d}s\right]\\
&\le (K_5+1)K_5\mathbb{E}\left[\int_{t_0}^T \left(1 + \delta^{1-\gamma}|W_s^{*,(t_0,w,\eta)}|^{1-\gamma}\right)\mathrm{d}s\right]\\
&\le (K_5+1)K_5\left(1 + \delta^{1-\gamma}\mathbb{E}\left[\sup_{s\in[t_0,T]} |W_s^{*,(t_0,w,\eta)}|^{1-\gamma}\right]\right)T < \infty.
\end{aligned}$$

Hence, the inequality (2.10) holds. Similarly, we have

$$\begin{aligned}
&\mathbb{E}_t\left[\int_t^T \delta e^{-\delta(s-t)}\log\left(\frac{C^*(W_s^{*,(t_0,w,\eta)})}{C^*(W_t^{*,(t_0,w,\eta)})}\right)^{1-\gamma}\mathrm{d}s + e^{-\delta(T-t)}\log\left(\frac{(1-\gamma)\overline{k}u_\gamma(W_T^{*,(t_0,w,\eta)})}{(C^*(W_t^{*,(t_0,w,\eta)}))^{1-\gamma}}\right)\right]\\
&= (1-\gamma)\mathbb{E}_t\left[\int_t^T \delta e^{-\delta(s-t)}\widehat{W}_{t,s}\mathrm{d}s + e^{-\delta(T-t)}\widehat{W}_{t,T}\right] + e^{-\delta(T-t)}\log\frac{\overline{k}}{\delta^{1-\gamma}}, \qquad \text{(A.23)}
\end{aligned}$$

for all $t \in [t_0, T]$, where

$$\widehat{W}_{t,s} := \log\left(\frac{W_s^{*,(t_0,w,\eta)}}{W_t^{*,(t_0,w,\eta)}}\right) = \int_t^s r_u^{*,(t_0,\eta)}\mathrm{d}u + M_s^{*,(t_0,\eta)} - M_t^{*,(t_0,\eta)} - \frac{1}{2}\int_t^s \left(\Pi^*(\eta_u^{(t_0,\eta)})\sigma(\eta_u^{(t_0,\eta)})\right)^2\mathrm{d}u$$
$$+ \int_t^s\int_{\mathbb{R}} \Big(\log\left(1+\Pi^*(\eta_u^{(t_0,\eta)})x\right) - \Pi^*(\eta_u^{(t_0,\eta)})x\Big)\widetilde{\mu}(\mathrm{d}u,\mathrm{d}x), \qquad s \in [t,T].$$

Similarly to the above, there exists a positive constant K_6 such that $\mathbb{E}_t[\sup_{s\in[t,T]}|\widehat{W}_{t,s}|] \le K_6 < \infty$ holds for any $t \in [t_0, T]$. Therefore, the following holds a.s.:

$$\left|(1-\gamma)\mathbb{E}_t\left[\int_t^T \delta e^{-\delta(s-t)}\widehat{W}_{t,s}\mathrm{d}s + e^{-\delta(T-t)}\widehat{W}_{t,T}\right]\right| \le |1-\gamma|K_6 < \infty, \qquad t \in [t_0, T]$$

From the above inequality and (A.23), we can see that the inequalities (2.8) and (2.13) hold. Hence, $(C^*(W^{*,(t_0,w,\eta)}), \overline{k}u_\gamma(W_T^{*,(t_0,w,\eta)})) \in \mathcal{C}^{\mathrm{U}}_{\gamma,T}$. In summary, $(C^*(W_t^{*,(t_0,w,\eta)}), \Pi^*(\eta_t^{(t_0,\eta)}))_{t\in[t_0,T]} \in \mathcal{A}(t_0, w, \eta)$ holds for any $(t_0, w, \eta) \in [0,T] \times (0,\infty) \times \mathbb{R}$.

Let $v_t^* = v(t, W_t^{*,(t_0,w,\eta)}, \eta_t^{(t_0,\eta)})$ and $(C_t^*, \Pi_t^*) = (C^*(W_t^{*,(t_0,w,\eta)}), \Pi^*(\eta_t^{(t_0,\eta)}))$ for $t \in [0,T]$. Similarly to the first part of this proof, we have

$$v_t^* = \overline{k}u_\gamma(W_T^{*,(t_0,w,\eta)}) + \int_t^T \left(HJB_s^{C_s^*,\Pi_s^*} + f(C_s^*, v_s^*)\right)\mathrm{d}s - \left(M_T^{*,\mathrm{loc.}} - M_t^{*,\mathrm{loc.}}\right), \qquad t \in [t_0, T],$$

where $M^{*,\mathrm{loc.}}$ is a local martingale on $[t_0,T]$. By the definition of (C^*, Π^*), $HJB_s^{C_s^*,\Pi_s^*} = 0$ holds for any $s \in [t_0, T]$ a.s. Thus,

$$v_t^* + \int_{t_0}^t f(C_s^*, v_s^*)\mathrm{d}s,$$

is a local martingale on $[t_0, T]$. Since $\mathbb{E}[\sup_{t\in[t_0,T]}|W_t^{*,(t_0,w,\eta)}|^{1-\gamma}] < \infty$ holds, $(v_t^*)_{t\in[t_0,T]}$ is uniformly integrable. Moreover, $v_t^*/u_\gamma(C_t^*) \ge \underline{h}\delta^{\gamma-1} > 0$ holds for any $t \in [t_0, T]$. Thus, from Proposition 2.5, $v_t \le U_t(C^*, \overline{k}u_\gamma(W_T^{*,(t_0,w,\eta)}))$ holds for any $t \in [t_0, T]$ a.s. Taking $t = t_0$ yields $v(t_0, w, \eta) \le U_{t_0}(C^*, \overline{k}u_\gamma(W_T^{*,(t_0,w,\eta)})) \le V(t_0, w, \eta)$. Combined with the previous inequality, $v = V$ holds and $(C^*(w), \Pi^*(\eta))$ is an admissible and optimal feedback control. □

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