Generalized Cobb-Douglas Production Functions and Aggregate Production Efficiency by a Monopoly

Tadashi HAMANO

Abstract

Increasing returns yields scale merits of production. Under these technologies monopoly may produce more output than two or more firms combined can do. However, increasing returns alone do not lead monopoly to achieve aggregate production efficiency. In this paper we consider an economy with one output, two inputs and two increasing returns technologies expressed by generalized Cobb-Douglas production functions and derive a sufficient condition for monopoly to lead aggregate production efficiency under increasing returns.

1. Introduction

In a convex environment, profit maximization of firms in a competitive market yields aggregate as well as individual production efficiency. That is, given prices, a profit maximizing production plan for each firm is on the frontier of the individual production set; and the aggregate production plan is on the frontier of the aggregate production set. So, decentralization results in aggregate production efficiency[1),2)].

However, under non-convex or increasing returns to scale technologies, decentralization may not lead to aggregate production efficiency. When more than one firm provide the same output, using increasing returns to scale technologies, marginal cost pricing equilibrium may lead to aggregate production inefficiency. In fact, Beato-Mas-Colell (1985) showed that, in an economy with one input and one output where there are two different types of technologies — one is constant returns to scale and the other is increasing returns to scale — none of marginal cost pricing equilibria achieves aggregate production efficiency. It should be noted that, due to the assumption of a single technology, the theory of natural monopoly is not applicable to the Beato-Mas-Colell's example in which technologies can be regarded as a kind of natural monopoly[3)].

Hamano (1996) considered an economy where there are many firms with increasing returns to scale technologies and attempted to derive sufficient conditions ensuring that monopoly achieves aggregate production efficiency. More specially, Hamano (1996) examined an economy with one output and many inputs where production technologies are expressed by production functions, and obtained such sufficient conditions which can be interpreted as non-decreasing "generalized" average productivity of inputs for each firm. Hamano (1996) also showed that a special class of generalized Cobb-Douglas production functions yields aggregate production efficiency by a monopoly.

In this paper we examine an economy with one output and two inputs where there are two production technologies expressed by generalized Cobb-Douglas production functions. We derive an alternative sufficient condition for monopoly to lead aggregate production efficiency. This condition may cover the case in which non-decreasing "generalized" average productivity of inputs does not hold. However, our result does not imply his condition even in our framework.

The organization of the paper is as follows: Section 2 presents a basic framework and a concept of an aggregate production function. Hamano's (1996) result is also explained. In Section 3 we derive a sufficient condition that monopoly achieves aggregate production efficiency in an economy where production technologies are expressed by Cobb-Douglas production functions. Section 4 provides proofs of Lemmas. In Section 5 we present some examples to compares our results with Hamano's (1996). Finally, we make some remarks on further research in Section 6.

2. Model and Review

Let us consider an economy with one output, two inputs and two firms. The technology of the h-th firm (h=1, 2) is represented by a production function f^h defined from R^2_+ to R_+. That is, given a vector of inputs $(x, y) \in R^2_+$, $f^h(x, y)$ is a maximum amount of output. For each h, a production function f^h is assumed to be non-decreasing and to satisfy the condition $f^h(0)=0$.

We recall the definition of a superadditive function[4].

Definition 1 *A function* $f : R^2_+ \to R_+$ *is called superadditive if, for all* (x, y) *and* (x', y'),

$$f(x + x', y + y') \geq f(x, y) + f(x', y').$$

We next give the definition of the aggregate production function at $(x,y) \in R^2_+$, given individual production functions.

Definition 2 *Given individual production function f^h $(h=1,\ 2)$, the aggregate production function $AF: R^2_+ \to R_+$ is defined as*

$$AF(x,y) \equiv \max_{\substack{(x,y)=(x^1,y^1)+(x^2,y^2) \\ (x^h,y^h)\in R^2_+\ (h=1,2)}} \left\{ f^1(x^1,y^1) + f^2(x^2,y^2) \right\}. \tag{1}$$

The following result is due to Hamano (1996), in which a sufficient condition for monopoly to achieve aggregate production efficiency is derived.

Proposition 1 (Hamano (1996)) *Suppose that there exists a superadditive function $l: R^2_+ \to R_+$ such that (i) $l(x,y) > 0$ for all $(x,y) \in \{(x,y) \in R^2_+ : f^h(x,y) > 0$ for some $h\}$; and that (ii) for all $h=1,2$, and for all (x,y) and (x',y') in $\{(x,y) \in R^m_+ : f^h(x,y) > 0\}$,*

$$\frac{f^h(x+x',y+y')}{l(x+x',y+y')} \geq \frac{f^h(x,y)}{l(x,y)}. \tag{2}$$

Then, we have

$$AF(x,y) = \max[f^1(x,y),\ f^2(x,y)], \tag{3}$$

for all $(x,y) \in R^2_+$.

Note that his result is proved in a general framework where both the number of inputs and that of firms may be more than two.

Throughout the paper we assume that production functions are expressed by generalized Cobb-Douglas types.

Assumption 1 *Production function of firm h is expressed by*

$$f^h(x,y) = A_h x^{\alpha_h} y^{\beta_h},$$

where $A_h,\ \alpha_h,\ \beta_h > 0$ and $\alpha_h + \beta_h \geq 1$ for $h=1,2$.

In Hamano (1996) the following result is derived as a corollary of Proposition 1

Corollary 1 *If $\min\{\alpha_1,\alpha_2\} + \min\{\beta_1,\beta_2\} \geq 1$ is satisfied, then we obtain the equality*

$$AF(x,y) = \max[A_1 x^{\alpha_1} y^{\beta_1},\ A_2 x^{\alpha_2} y^{\beta_2}],$$

for all $(x,y) \in R^2_+$.

3. Main Result

In the previous section we refer to in our framework a sufficient condition for monopoly to lead aggregate production efficiency. We now provide an alternative sufficient condition.

Theorem 1 *Let us define* α *and* β *as follows:*

$$\alpha = \min(\alpha_1, \beta_2), \tag{4}$$
$$\beta = \min(\alpha_2, \beta_1). \tag{5}$$

If $\alpha + \beta < 4\alpha\beta$, *then the following equality holds for all* $(x,y) \in R^2_+$,

$$AF(x,y) = \max[A_1 x^{\alpha_1} y^{\beta_1}, A_2 x^{\alpha_2} y^{\beta_2}].$$

To show Theorem 1 we shall use the following two lemmas, which will be proved in Section 4.

Lemma 1 *Let* $k^1(x,y) = A_1 x^{\alpha} y^{\beta}$ *and* $k^2(x,y) = A_2 x^{\alpha+\alpha'} y^{\beta+\beta'}$ *where* $A_1, A_2 > 0,\ \alpha, \beta > 0$ *and* $\alpha', \beta' \geq 0$. *If* $k^1(\tilde{x},\tilde{y}) = k^2(\tilde{x},\tilde{y})$ *for some* $(\tilde{x},\tilde{y})$, *then* $k^1(\tilde{x}+x, \tilde{y}+y) \leq k^2(\tilde{x}+x, \tilde{y}+y)$ *for all* $(x,y) \in R^2_+$.

Lemma 2 *Consider two symmetric functions:*

$$f^1(x,y) = A_1 x^{\alpha} y^{\beta},$$
$$f^2(x,y) = A_2 x^{\beta} y^{\alpha}.$$

If $\alpha + \beta < 4\alpha\beta$, *then for all* $(x,y) \in R^2_+$ *with* $0 \leq x \leq \bar{x}$ *and* $0 \leq y \leq \bar{y}$,

$$A_1 x^{\alpha} y^{\beta} + A_2 (\bar{x}-x)^{\beta} (\bar{y}-y)^{\alpha} \leq \max(A_1 \bar{x}^{\alpha} \bar{y}^{\beta}, A_2 \bar{x}^{\beta} \bar{y}^{\alpha}). \tag{6}$$

We now proceed to the proof of Theorem 1.

Proof of Theorem 1 It suffices to show that, given a vector of aggregate inputs $(\bar{x}, \bar{y}) \in R^2_+$, the following inequality holds for all $(x,y) \in R^2_+$ with $0 \leq x \leq \bar{x}$ and $0 \leq y \leq \bar{y}$,

$$A_1 x^{\alpha_1} y^{\beta_1} + A_2(\bar{x}-x)^{\alpha_2}(\bar{y}-y)^{\beta_2} \leq \max(A_1\bar{x}^{\alpha_1}\bar{y}^{\beta_1}, A_2\bar{x}^{\alpha_2}\bar{y}^{\beta_2}).$$

Suppose, on the contrary, that there exists $(\tilde{x},\tilde{y})$ such that

$$A_1 \tilde{x}^{\alpha_1} \tilde{y}^{\beta_1} + A_2(\bar{x}-\tilde{x})^{\alpha_2}(\bar{y}-\tilde{y})^{\beta_2} > \max(A_1\bar{x}^{\alpha_1}\bar{y}^{\beta_1}, A_2\bar{x}^{\alpha_2}\bar{y}^{\beta_2}).$$

Now, set $\bar{A}_1$ and $\bar{A}_2$ as follows:

$$\begin{aligned}
\bar{A}_1 &= \frac{A_1\tilde{x}^{\alpha_1}\tilde{y}^{\beta_1}}{\tilde{x}^{\alpha}\tilde{y}^{\beta}}, \\
\bar{A}_2 &= \frac{A_2(\bar{x}-\tilde{x})^{\alpha_2}(\bar{y}-\tilde{y})^{\beta_2}}{(\bar{x}-\tilde{x})^{\beta}(\bar{y}-\tilde{y})^{\alpha}}.
\end{aligned}$$

Then, Lemma 2 yields

$$\bar{A}_1\tilde{x}^{\alpha}\tilde{y}^{\beta} + \bar{A}_2(\bar{x}-\tilde{x})^{\beta}(\bar{y}-\tilde{y})^{\alpha} \leq \max\left(\bar{A}_1\bar{x}^{\alpha}\bar{y}^{\beta}, \bar{A}_2\bar{x}^{\beta}\bar{y}^{\alpha}\right).$$

Since $\alpha = \min\,(\alpha_1,\beta_2)$ and $\beta = \min\,(\alpha_2,\beta_1)$, it also follows from Lemma 1 that

$$\begin{aligned}
&\bar{A}_1\bar{x}^{\alpha}\bar{y}^{\beta} \leq A_1\bar{x}^{\alpha_1}\bar{y}^{\beta_1}, \\
&\bar{A}_2\bar{x}^{\beta}\bar{y}^{\alpha} \leq A_2\bar{x}^{\alpha_2}\bar{y}^{\beta_2}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
A_1 \tilde{x}^{\alpha_1} \tilde{y}^{\beta_1} + A_2(\bar{x}-\tilde{x})^{\alpha_2}(\bar{y}-\tilde{y})^{\beta_2} &= \bar{A}_1\tilde{x}^{\alpha}\tilde{y}^{\beta} + \bar{A}_2(\bar{x}-\tilde{x})^{\beta}(\bar{y}-\tilde{y})^{\alpha} \\
&\leq \max\left(\bar{A}_1\bar{x}^{\alpha}\bar{y}^{\beta}, \bar{A}_2\bar{x}^{\beta}\bar{y}^{\alpha}\right) \\
&\leq \max\left(A_1\bar{x}^{\alpha_1}\bar{y}^{\beta_1}, A_2\bar{x}^{\alpha_2}\bar{y}^{\beta_2}\right),
\end{aligned}$$

a contradiction. **Q.E.D.**

4. Proofs of Lemmas

4.1 Proof of Lemma 1

Since $k^1(\tilde{x},\tilde{y}) = k^2(\tilde{x},\tilde{y})$, we have $A_1\tilde{x}^{\alpha}\tilde{y}^{\beta} = A_2\tilde{x}^{\alpha+\alpha'}\tilde{y}^{\beta+\beta'}$, which implies that

$$\frac{A_2}{A_1} = \frac{1}{\tilde{x}^{\alpha'}\tilde{y}^{\beta'}}.$$

Now, from the condition $\alpha', \beta' \geq 0$, it follows that, for all $(x,y) \in R^2_+$

$$\begin{aligned}
k^2(\tilde{x}+x,\tilde{y}+y) &= A_1(\tilde{x}+x)^{\alpha}(\tilde{y}+y)^{\beta} \times \frac{A_2}{A_1}(\tilde{x}+x)^{\alpha'}(\tilde{y}+y)^{\beta'} \\
&= k^1(\tilde{x}+x,\tilde{y}+y) \times \frac{(\tilde{x}+x)^{\alpha'}(\tilde{y}+y)^{\beta'}}{\tilde{x}^{\alpha'}\tilde{y}^{\beta'}} \geq k^1(\tilde{x}+x,\tilde{y}+y).
\end{aligned}$$

Q.E.D.

4.2 Proof of Lemma 2

We can assume without loss of generality $\alpha \geq \beta$. Let a function F: $R_+^2 \rightarrow R$ be defined by

$$F(x,y) = A_1 x^\alpha y^\beta + A_2(\bar{x} - x)^\beta (\bar{y} - y)^\alpha. \tag{7}$$

Let (x^*, y^*) be the point in R^2 at which the function $F\,(x,y)$ achieves the (global) maximum. We also define a set $\bar{\Omega} \subset R_+^2$ as follows:

$$\bar{\Omega} \equiv \{(x,y) \in R_+^2 :\ 0 \leq x \leq \bar{x} \ \text{ and } \ 0 \leq y \leq \bar{y}\}.$$

If the function $F\,(x,y)$ does not achieve the maximum in the interior of $\bar{\Omega}$, then it achieves the maximum at the boundary of $\bar{\Omega}$. In this case it is clear that either $(x^*, y^*) =$ (0,0) or $(x^*, y^*) = (\bar{x}, \bar{y})$ holds. This implies that the inequality (6) is true.

In order to show that $F\,(x,y)$ does not achieve the maximum in the interior of $\bar{\Omega}$, it suffices to show that the second order condition (S.O.C.) for the local maximum does not hold at the points $(\hat{x}, \hat{y})$ satisfying the first order condition (F.O.C.) in the interior. In the remaining of the proof we shall show that

$$F_{xx}(\hat{x},\hat{y})F_{yy}(\hat{x},\hat{y}) - (F_{xy}(\hat{x},\hat{y}))^2 < 0.$$

Partial differentiations of $F(x,y)$ give:

$$F_x(x,y) = \frac{\partial F(x,y)}{\partial x} = A_1 \alpha x^{\alpha-1} y^\beta - A_2 \beta (\bar{x} - x)^{\beta-1} (\bar{y} - y)^\alpha,$$
$$F_y(x,y) = \frac{\partial F(x,y)}{\partial y} = A_1 \beta x^\alpha y^{\beta-1} - A_2 \alpha (\bar{x} - x)^\beta (\bar{y} - y)^{\alpha-1}.$$

Thus, F.O.C. can be expressed as follows:

$$A_1 \alpha x^{\alpha-1} y^\beta = A_2 \beta (\bar{x} - x)^{\beta-1} (\bar{y} - y)^\alpha, \tag{8}$$
$$A_1 \beta x^\alpha y^{\beta-1} = A_2 \alpha (\bar{x} - x)^\beta (\bar{y} - y)^{\alpha-1}. \tag{9}$$

Equations (8) and (9) imply

$$\frac{\frac{\alpha}{x}}{\frac{\beta}{y}} = \frac{\frac{\beta}{\bar{x}-x}}{\frac{\alpha}{\bar{y}-y}},$$

which yields

$$y = \frac{\beta^2 \bar{y} x}{\alpha^2 \bar{x} + (\beta^2 - \alpha^2) x}. \tag{10}$$

Second order partial differentiation of $\boldsymbol{F}(x,y)$ also gives:

$$F_{xx}(x,y)=A_1\alpha(\alpha-1)x^{\alpha-2}y^{\beta}+A_2\beta(\beta-1)(\bar{x}-x)^{\beta-2}(\bar{y}-y)^{\alpha}, \tag{11}$$
$$F_{yy}(x,y)=A_1\beta(\beta-1)x^{\alpha}y^{\beta-2}+A_2\alpha(\alpha-1)(\bar{x}-x)^{\beta}(\bar{y}-y)^{\alpha-2}, \tag{12}$$

$$\begin{aligned} F_{xy}(x,y)&=A_1\alpha\beta x^{\alpha-1}y^{\beta-1}+A_2\alpha\beta(\bar{x}-x)^{\beta-1}(\bar{y}-y)^{\alpha-1}\\ &=\alpha\beta\{A_1x^{\alpha-1}y^{\beta-1}+A_2(\bar{x}-x)^{\beta-1}(\bar{y}-y)^{\alpha-1}\}. \end{aligned} \tag{13}$$

Now, using the equation (8) of F.O.C., we rewrite the expression (11) as follows:

$$F_{xx}(x,y)=A_1\alpha x^{\alpha-2}y^{\beta}(\bar{x}-x)^{-1}\{(\alpha-1)\bar{x}-(\alpha-\beta)x\}. \tag{14}$$

Similarly, using the equation (9) of F.O.C., we rewrite the expression (12) as follows:

$$F_{yy}(x,y)=A_1\beta x^{\alpha}y^{\beta-2}(\bar{y}-y)^{-1}\{(\beta-1)\bar{y}+(\alpha-\beta)y\}. \tag{15}$$

For $F_{xy}(x,y)$, we rewrite the expression (13) in two different forms. If we use the equation (8), then the expression (13) can be expressed as

$$F_{xy}(x,y)=A_1\alpha x^{\alpha-1}y^{\beta-1}(\bar{y}-y)^{-1}\{\beta\bar{y}+(\alpha-\beta)y\}. \tag{16}$$

If we use the equation (9), then the expression (13) can be expressed as

$$F_{xy}(x,y)=A_1\beta x^{\alpha-1}y^{\beta-1}(\bar{x}-x)^{-1}\{\alpha\bar{x}-(\alpha-\beta)x\}. \tag{17}$$

Let us now calculate the value of $F_{xx}(x,y)F_{yy}(x,y)-(F_{xy}(x,y))^2$. It follows from expressions (14)−(17) with (10) that

$$\begin{aligned} &F_{xx}(x,y)F_{yy}(x,y)-(F_{xy}(x,y))^2\\ &=A_1^2\alpha x^{\alpha-2}y^{\beta}(\bar{x}-x)^{-1}\{(\alpha-1)\bar{x}-(\alpha-\beta)x\}\\ &\qquad\qquad\times A_1\beta x^{\alpha}y^{\beta-2}(\bar{y}-y)^{-1}\{(\beta-1)\bar{y}+(\alpha-\beta)y\}\\ &\quad-A_1\alpha x^{\alpha-1}y^{\beta-1}(\bar{y}-y)^{-1}\{\beta\bar{y}+(\alpha-\beta)y\}\\ &\qquad\qquad\times A_1\beta x^{\alpha-1}y^{\beta-1}(\bar{x}-x)^{-1}\{\alpha\bar{x}-(\alpha-\beta)x\}\\ &=A_1^2\alpha\beta x^{2\alpha-2}y^{2\beta-2}(\bar{x}-x)^{-1}(\bar{y}-y)^{-1}\\ &\quad\times\begin{bmatrix}\{(\alpha-1)\bar{x}-(\alpha-\beta)x\}\times\{(\beta-1)\bar{y}+(\alpha-\beta)y\}\\ -\{\alpha\bar{x}-(\alpha-\beta)x\}\times\{\beta\bar{y}+(\alpha-\beta)y\}\end{bmatrix}\\ &=A_1^2\alpha\beta x^{2\alpha-2}y^{2\beta-2}(\bar{x}-x)^{-1}(\bar{y}-y)^{-1}\\ &\quad\times\frac{\bar{y}}{\alpha^2\bar{x}-(\alpha^2-\beta^2)x}\times\begin{bmatrix}(1-\alpha-\beta)\{\alpha^2\bar{x}-(\alpha^2-\beta^2)x\}\bar{x}\\ +(\alpha-\beta)\{\alpha^2\bar{x}-(\alpha^2-\beta^2)x\}x\\ -(\alpha-\beta)\beta^2\bar{x}x\end{bmatrix}. \end{aligned} \tag{18}$$

We note that the sign of expressions outside the square brackets in (18) is positive. Thus, setting the expressions inside the square brackets in (18) as S, then we have

$$\begin{aligned} S \equiv & (1-\alpha-\beta)\{\alpha^2\bar{x}-(\alpha^2-\beta^2)x\}\bar{x} \\ & \qquad + (\alpha-\beta)\{\alpha^2\bar{x}-(\alpha^2-\beta^2)x\}x-(\alpha-\beta)\beta^2\bar{x}x \\ = & -(\alpha-\beta)(\alpha^2-\beta^2)x^2 \\ & \qquad + (\alpha-\beta)(\alpha+\beta)(2\alpha-1)\bar{x}x+(1-\alpha-\beta)\alpha^2\bar{x}^2. \end{aligned} \tag{19}$$

If we set the right hand side of (19) to zero, then we have the quadratic equation:

$$-(\alpha-\beta)(\alpha^2-\beta^2)x^2+(\alpha-\beta)(\alpha+\beta)(2\alpha-1)\bar{x}x+(1-\alpha-\beta)\alpha^2\bar{x}^2=0. \tag{20}$$

Now, the discriminant Δ of this quadratic equation (20) of a variable x can be expressed as follows:

$$\begin{aligned} \Delta &= (\alpha-\beta)^2(\alpha+\beta)^2(2\alpha-1)^2\bar{x}^2+4(\alpha-\beta)(\alpha^2-\beta^2)(1-\alpha-\beta)\alpha^2\bar{x}^2 \\ &= (\alpha-\beta)^2(\alpha+\beta)\bar{x}^2\{(\alpha+\beta)(2\alpha-1)^2+4\alpha^2(1-\alpha-\beta)\} \\ &= (\alpha-\beta)^2(\alpha+\beta)\bar{x}^2(\alpha+\beta-4\alpha\beta). \end{aligned}$$

If $\alpha+\beta-4\alpha\beta<0$ or $\alpha+\beta<4\alpha\beta$, then $\Delta<0$. Since the coefficient of x^2 is negative, the quadratic equation (20) has no real roots. This implies that $S<0$ for all $x \in [0,\bar{x}]$. Therefore, we conclude that, if $\alpha+\beta<4\alpha\beta$ holds, then

$$F_{xx}(\hat{x},\hat{y})F_{yy}(\hat{x},\hat{y})-(F_{xy}(\hat{x},\hat{y}))^2<0$$

for all $(\hat{x},\hat{y})$ satisfying the first order conditions (8) and (9). **Q.E.D.**

5. Examples

In this section we provide two examples in both of which monopoly production leads to aggregate production efficiency. We first show that there are two Cobb-Douglas production functions which satisfies our condition (Theorem 1) but not Hamano's (1996) (Corollary 1).

Example 1 *Let us consider the following two Cobb-Douglas production functions:*

$$\begin{aligned} f^1(x,y) &= xy^{0.34}, \\ f^2(x,y) &= x^{0.34}y. \end{aligned}$$

Now, set α = min (1,1) = 1 *and* β = min (0.34, 0.34) = 0.34. *Then,* α *and* β *satisfy the condition of Theorem 1. However, the condition of Corollary 1 does not hold in this example.*

The next example is two Cobb-Douglas production functions which satisfies Hamano's (1996) but not ours.

Example 2 *We consider two Cobb-Douglas production functions:*

$$f^1(x,y) = xy^{1/4},$$
$$f^2(x,y) = x^{3/4}y.$$

Now, set α = min (1, 1) = 1 *and* β = min (1/4, 3/4) =1/4. *Then, this example satisfies the condition of Corollary 1. However, the condition of Theorem1 does not hold in this example.*

In sum these examples illustrate that Hamano's (1996) result does not imply ours nor ours does Hamano's.

6. Concluding Remark

In this paper we present a sufficient condition for monopoly to lead aggregate production efficiency in an economy with one output, two inputs and two increasing returns technologies expressed by Cobb-Douglas production functions.

Our framework is restricted in several ways. First of all we can not handle cases of economies with more than two inputs or technologies. It is not straightforward to extend our result to more general framework. Second, since our proof heavily depends upon functional forms specified by Cobb-Douglas production functions. It seems difficult to extend our results to those cases without such specification.

Finally, Corollary1 or Theorem 1 give sufficient conditions for efficient production by a monopoly. It is unclear whether decentralization is better than monopoly if those conditions do not hold. This question is still left open even in our restricted setting.

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Notes

1) See Mas-Colell et al. (1995, pp. 147-149).
2) We do not deal with aggregation problems in macroeconomics; for this issue, see Felipe and Fisher (2003), for example.
3) See Sharkey (1982) for the theory of natural monopoly.
4) See Rosenbaum (1950).

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